Communications and Control Engineering

Giuseppe Conte · Claude H. Moog
Anna Maria Perdon

Algebraic Methods for Nonlinear Control Systems

Theory and Applications

2nd Edition

Springer

Communications and Control Engineering

Published titles include:

Stability and Stabilization of Infinite Dimensional Systems with Applications
Zheng-Hua Luo, Bao-Zhu Guo and Omer Morgul

Nonsmooth Mechanics (Second edition)
Bernard Brogliato

Nonlinear Control Systems II
Alberto Isidori

L_2-Gain and Passivity Techniques in Nonlinear Control
Arjan van der Schaft

Control of Linear Systems with Regulation and Input Constraints
Ali Saberi, Anton A. Stoorvogel and Peddapullaiah Sannuti

Robust and H_∞ Control
Ben M. Chen

Computer Controlled Systems
Efim N. Rosenwasser and Bernhard P. Lampe

Control of Complex and Uncertain Systems
Stanislav V. Emelyanov and Sergey K. Korovin

Robust Control Design Using H_∞ Methods
Ian R. Petersen, Valery A. Ugrinovski and Andrey V. Savkin

Model Reduction for Control System Design
Goro Obinata and Brian D.O. Anderson

Control Theory for Linear Systems
Harry L. Trentelman, Anton Stoorvogel and Malo Hautus

Functional Adaptive Control
Simon G. Fabri and Visakan Kadirkamanathan

Positive 1D and 2D Systems
Tadeusz Kaczorek

Identification and Control Using Volterra Models
Francis J. Doyle III, Ronald K. Pearson and Bobatunde A. Ogunnaike

Non-linear Control for Underactuated Mechanical Systems
Isabelle Fantoni and Rogelio Lozano

Robust Control (Second edition)
Jürgen Ackermann

Flow Control by Feedback
Ole Morten Aamo and Miroslav Krstić

Learning and Generalization (Second edition)
Mathukumalli Vidyasagar

Constrained Control and Estimation
Graham C. Goodwin, María M. Seron and José A. De Doná

Randomized Algorithms for Analysis and Control of Uncertain Systems
Roberto Tempo, Giuseppe Calafiore and Fabrizio Dabbene

Switched Linear Systems
Zhendong Sun and Shuzhi S. Ge

Subspace Methods for System Identification
Tohru Katayama

Digital Control Systems
Ioan D. Landau and Gianluca Zito

Multivariable Computer-controlled Systems
Efim N. Rosenwasser and Bernhard P. Lampe

Dissipative Systems Analysis and Control (Second edition)
Bernard Brogliato, Rogelio Lozano, Bernhard Maschke and Olav Egeland

Polynomial and Rational Matrices
Tadeusz Kaczorek

Giuseppe Conte, Claude H. Moog and
Anna Maria Perdon

Algebraic Methods for Nonlinear Control Systems

2nd Edition

Springer

Giuseppe Conte, Professor
Università Politecnica delle Marche
Dipartimento di Ingegneria Informatica
Gestionale e dell'Automazione
60131 Ancona
Italy

Claude H. Moog, Director of Research
Centre National de la Recherche
Scientifique (CNRS)
Institut de Recherche en Communications
et en Cybernétique de Nantes (IRCCyN)
44321 Nantes Cedex 3
France

Anna Maria Perdon, Professor
Università Politecnica delle Marche
Dipartimento di Ingegneria Informatica
Gestionale e dell'Automazione
60131 Ancona
Italy

British Library Cataloguing in Publication Data
Conte, G., 1951-
Algebraic methods for nonlinear control systems. - 2nd ed.
- (Communications and control engineering)
1. Nonlinear control theory
I. Title II. Moog, C. H., 1955- III. Perdon, A. M. IV.
Conte, G., 1951-. Nonlinear control systems
629.8'36
ISBN-13: 9781846285943
ISBN-10: 1846285941

Library of Congress Control Number: 2006936018

Communications and Control Engineering Series ISSN 0178-5354
ISBN 978-1-84628-594-3 2nd edition e-ISBN 1-84628-595-X 2nd edition Printed on acid-free paper
ISBN 1-85233-151-8 1st edition

First published 1999, Second edition 2007

Previously published as *Nonlinear Control Systems* in Springer's *Lecture Notes in Control and Information Sciences* series

9 8 7 6 5 4 3 2 1

Springer Science+Business Media
springer.com

To Federica Paola

To Renan-Abhinav

Preface to the Second Edition

The first edition of this book appeared at the end of the last century and, since then, it has been used as a textbook in several graduate courses, summer schools, and a preconference tutorial workshop on Nonlinear Systems. Thanks to the experience gained in these activities, several modifications appeared to be appropriate. The chapter on modeling was enlarged and it now includes results on standard realization of nonlinear systems. This is motivated by the importance of state-space representations in the analysis and synthesis of nonlinear control systems in the current literature. The focus of the book remained about structural properties that do not involve stability issues and, for this reason, those issues are only marginally considered. The chapter on systems structure has been enlarged by adding more material on system inversion and by adding motivational examples. A new chapter on output feedback has been included at the end of the book, in view of the major importance it has in practical applications. Various supporting practical examples borrowed from robotics, mechanics, and other application areas have been added throughout the book. A few exercises complete the chapters.

The introduction to the differential algebraic approach has been deleted because new monographs on this topic are available.

Finally, solutions to the problems can be found on the following website: http://www.springer.com/1-84628-594-1

Ancona, May 2006, *Giuseppe Conte and Annamaria Perdon*
Nantes, May 2006, *Claude H. Moog*

Preface to the First Edition

The theory of nonlinear control systems owes a large part of its modern development and success to the systematic use of differential geometric methods and tools. One of the first problems to be considered from the point of view of differential geometry was, at the beginning of the 1970s, that of analyzing the controllability of a nonlinear system. Early works on that topic ([111, 154, 155, 9]) highlighted the power and the potentiality of the differential geometric approach and motivated the interest of many researchers.

During the 1980s, the possibilities offered by the use of differential geometric techniques in the study of nonlinear control systems were largely exploited. One of the underlying leading ideas (see [75, 87]) was that of generalizing, to the greatest possible extent, the so-called geometric approach which had been first developed in the linear case (see [6, 160]). The research effort produced, in that period, many important results and it provided effective solutions to several control problems, such as disturbance decoupling problems, noninteracting control problems, and model matching problems. Excellent and comprehensive descriptions of the methodology and of the results achieved, together with meaningful examples of applications, can be found in [86] and in [126].

In the second half of the 1980s the limits of the differential geometric approach started to be explored and to become known. In particular, it became clear that problems such as system inversion or the synthesis of dynamic feedbacks could hardly be tackled with the already well-established differential geometric methods.

In the same period, the introduction of differential algebraic methods in the study of nonlinear control systems ([49]) offered a way to circumvent a number of difficulties encountered up to that time. The use of differential algebraic concepts characterized, through the work of several authors, a novel approach, which has essentially an algebraic nature, and, at the same time, it provided additional tools for investigating old and new problems. Further results on problems pertaining to inversion, noninteracting control, realization and reduction to canonical forms were obtained in the following years, and others were made achievable.

Today, the use of an algebraic point of view in nonlinear control problems has gained popularity and diffusion. This motivates the present book, whose aim is to give an account of the algebraic approach to nonlinear system theory and of its development in recent years. Together with a number of results which are scattered in the literature, the reader will find in it a self contained, comprehensive description of techniques and tools that can enrich his equipment as a control theorist and can provide a solution to otherwise not easily tractable control problems.

One of the distinctive characteristics that makes the algebraic approach interesting and useful is its inherent simplicity. In comparison with the mathematical background needed for profitably employing differential geometric methods, the knowledge required for using the tools described in this book is very limited. A significant example of this is offered by the way in which the notion of accessibility and the problem of linearization are dealt with. In both cases, a single tool, based on elementary differentiation of a function, namely, the notion of relative degree, gives the key for carrying on a deep analysis and for characterizing relevant dynamic properties. From a didactic point of view, simplicity renders the algebraic approach a practicable and valid choice in teaching engineering courses on nonlinear control. The book emphasizes this aspect and is usable as a teaching aid. In addition, simplicity facilitates the development of efficient algorithmic procedures that are relevant in solving concrete analysis and synthesis problems.

Another positive quality of the algebraic approach is its wide applicability in the field of dynamic systems and control. Although only continuous-time systems are considered in the book, the tools and methods described apply successfully to a number of control problems concerning discrete-time nonlinear systems, as shown in [4, 68]. Applications to time-varying systems are also possible, and recent results have been obtained in dealing with time-delay systems (see [13, 120]). With respect to other general methodologies, then, the algebraic approach appears to be more versatile and capable of going to the heart of the problem.

Only a basic knowledge of systems and control theory is required for reading the book, whose material is arranged in a self-explicatory way. The general setting and the fundamental notions are described and illustrated in the first part, entitled Methodology. Mathematical preliminaries are presented including notations from exterior differentiation. The system analysis completes this part and deals with fundamental properties as accessibility and observability. The structure algorithm and a canonical decomposition of the system are given as well.

In the second part, entitled Applications to Control Problems, the tools and techniques of the algebraic approach are employed for solving a number of basic control problems that are of practical interest in fields such as robotics and control of general mechanical systems, as well as in process control. The solution of the feedback linearization problem is given in terms of the accessibility filtration $\{\mathcal{H}_k\}$ introduced in Chapter 3. The disturbance decoupling problem is solved using the subspace $\mathcal{X} \cap \mathcal{Y}$, namely, the subspace that is observable independently from the input (Chapter 4). In the noninteracting control problem and in the model matching problem, we use the output filtration $\{\mathcal{E}_k\}$ and the structure algorithm (Chapter 5).

Finally, in the third part, entitled Differential Algebra, differentially algebraic tools and concepts are introduced in an elementary, but comprehensive, way, and the results of the first parts are revisited and analyzed from a differentially algebraic point of view. The notions described in the third part may contribute to expand not only the technical knowledge of the reader, but also his comprehension of the key ideas of the algebraic approach. A conceptually powerful way of approaching the theory of nonlinear systems has been proposed at the end of the 1980s in [49, 52, 54], introducing the use of differential algebra and differential-algebraic methods. In comparison with other approaches which employ differential geometric methods (see [86, 126] for a comprehensive description) or Volterra or generating series (see [48]), this one appears in particular capable of removing some drawbacks present in the notions of rank and it allows us to characterize general invertible compensators. In addition, it provides useful insight and results in connection with various analysis and synthesis problems (see [44, 52, 54, 64, 136]). Although differential algebraic methods are better suited for dealing with systems described by polynomials or rational functions, extensions to more general cases (as suggested, for instance, in [50, 141]) are possible. Here, we give a brief introduction to the principal notions of differential algebra and we introduce a general notion of dynamic systems in differential algebraic terms. Related system theoretical properties are described and the principal results obtained by the differential algebraic approach are mentioned without entering into the detailsto help the reader in establishing a connection with the notions studied in the previous parts of this book.

The authors acknowledge the NATO for its financial support of a joint research project under grant CRG 890101. Several results of this project are reported in this book.

Finally, the authors would like to thank J.W. Grizzle, and M.D. di Benedetto for some joint research work which inspired this book. Valuable discussions with M. Fliess, A. Isidori, A. Glumineau, E. Aranda, J.B. Pomet, R. Andiarti, Ü. Kotta, and Y.F. Zheng are acknowledged as well as the careful reading by L.A. Márquez Martínez and R. Pothin, and the help of E. Le Carpentier.

Ancona, December 1998, *Giuseppe Conte and Annamaria Perdon*
Nantes, December 1998, *Claude H. Moog*

Contents

Part I Methodology

Part I

Methodology

1

Preliminaries

In very general terms, a dynamic system is a mathematical object that models, in some way, the evolution over time of a given physical phenomenon. Many authors, in the recent history of system and control theory, have provided concrete refinements and specializations of the above informal definition, from the axiomatic characterization given in [99], to the development of the so-called behavioral point of view [133, 159], or to the module theoretical characterization proposed in [54].

Here, we do not enter into this debate, but, assuming the classical control theoretical point of view, we view dynamic systems as objects described by a system of first-order differential equations of the form

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1.1}$$

where the independent variable $t \in \mathbb{R}$ denotes time; the state $x(\cdot)$ belongs to $\mathbb{R}^n$; the input $u(\cdot)$ belongs to $\mathbb{R}^m$; the input $y(\cdot)$ belongs to $\mathbb{R}^p$; and the entries of f, g, h are functions in a sufficiently general class.

Dynamic systems of the above kind arise naturally in modeling, at least locally, many physical phenomena by first principles. Moreover, techniques based on these models have proved quite effective in analyzing and controlling objects such as machines, robots, vehicles as well as industrial, economic, and biological processes [101, 147]. These considerations respond to two primary concerns about the choice of our models, namely, generality and usefulness. However, to make the picture more precise, it remains to specify in which class of functions f, g, and h are taken. The following discussion will help us to make a motivated choice.

Modeling involves approximation and some degree of uncertainty in dealing with dynamic systems, we are naturally interested in properties whose validity in nominal situations may imply validity in almost all situations, that is, in almost all situations except, so to say, in pathological ones. A way to capture this feature in mathematical terms is that of considering *generic* properties, that is, properties that hold on open and dense subsets of suitable domains of

definition, provided they hold at some point of such domains. This, however, imposes some restriction on the class of mathematical objects we want to deal with because the concept of generic properties must make sense for them.
To understand this fact and its consequences better, let us consider, for instance, the role of the vector field $g(x)$ in (1.1). Using a model of the form (1.1), we may clearly ask that independent inputs have, in general, independent effects on the system. Otherwise, simpler models, in which the dimension of the input vector u is reduced, could be used, at least locally. Since this fact depends on the rank of the vector field $g(x)$ at different points x of the space of states, it is useful to require that the property of having maximal rank is generic for $g(x)$. In particular, this implies that vector fields $g(x)$ whose components are C^∞ functions are not admissible in (1.1), since there are C^∞ functions, e.g., $f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x < 0 \\ 0, & \text{if} x \geq 0 \end{cases}$, that, being neither generically zero nor generically different from zero, could give rise to vector fields whose rank is neither generically maximal nor generically lower than the maximum. In other terms, the notion of generic property does not make sense, in general, for systems defined by C^∞ functions. The situation is different if we restrict our attention to systems defined by analytic functions, and also meromorphic ones, and this, as stated in Section 1.2, motivates our choice throughout the book.

1.1 Analytic and Meromorphic Functions

To specify the class of functions we will deal with in our models, let us introduce the following definition.

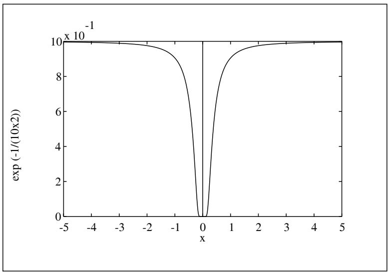

Fig. 1.1. Graph of $\exp(-1/10x^2)$

Definition 1.1. *Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \rightarrow \mathbb{R}$ is analytic at a point $x_0 \in I$ if it admits a Taylor series expansion in a neighborhood of x_0. If f is analytic at every point of $I \subseteq \mathbb{R}$, we say that f is analytic in I.*

Examples of analytic functions are given by polynomial functions, as well as by the trigonometric functions $\sin x$ and $\cos x$. Rational functions are analytic at any point in their domain of definition.
The function $f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases}$ is C^∞, but it is not analytic at $x = 0$ (see Figure 1.1).

The basic property of analytic functions we are interested in is stated in the following proposition.

Proposition 1.2. *Let $I \subseteq I\!R$ be an open interval, and let $f : I \to I\!R$ be an analytic function on I; then either*

(i) $f \equiv 0$ in I, or
(ii) the zeros of f in I are isolated.

Proof. Let $Z(f)$ denote the set of zeros of f in I. Since f is analytic in I, for every point $\bar{x} \in Z(f)$, there exists a neighborhood $D(\bar{x}, r) = \{x \in I\!R$ such that $\mid x - \bar{x} \mid < r\} \subseteq I$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - \bar{x})^n$$

for $x \in D(\bar{x}, r)$. Now, two cases are possible: either $c_n = 0$ for $n = 0, 1, 2, \ldots$, or there exists a minimal positive integer m such that $c_m \neq 0$ and $c_n = 0$ for $n < m$. In the latter case, we can write

$$f(x) = (x - \bar{x})^m f_1(x) \tag{1.2}$$

where

$$f_1(x) = \sum_{n=0}^{\infty} c_{n+m} (x - \bar{x})^n \text{ and} f_1(\bar{x}) \neq 0 \tag{1.3}$$

By continuity, $f_1(x) \neq 0$ in a neighborhood $D(\bar{x}, r_1)$ and also $f(x) \neq 0$ for $x \neq \bar{x}$ in $D(\bar{x}, r_1)$, then $\bar{x}$ is an isolated zero. Alternatively, in the first case, $f \equiv 0$ in $D(\bar{x}, r)$ and, hence, $\bar{x}$ is an interior point of $Z(f)$. Then, $Z(f)$ consists of points that are either isolated or interior. In particular, assume that there exists at least one point $\bar{x}$ in $Z(f)$ which is interior and let A be the connected component of $Z(f)$ that contains $\bar{x}$. If the point $sup\{A\}$ belongs to I, by continuity of f, it belongs to $Z(f)$ and hence, since it is not isolated but, necessarily, interior, it must coincide with $sup\{I\}$. Coincidence obviously holds also if $sup\{A\}$ does not belong to I. Then, we get $sup\{A\} = sup\{I\}$ and, in the same way, $inf\{A\} = inf\{I\}$. Therefore, $Z(f) = I$ and $f \equiv 0$ in I. ∎

A polynomial function has a finite number of zeros which are isolated. The function $f(x) = \sin x$ has an infinite number of isolated zeros located at $x = k\pi$ for any positive or negative integer k. A typical example of a nonanalytic continuous function whose zeros are not isolated is the following.

Example 1.3. The function $f(x)$, defined by

$$f(x) = \begin{cases} \sin(1/x), & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases}$$

is not analytic since $x = 0$ is a point of accumulation for the zeros of f (see Figure 1.3).

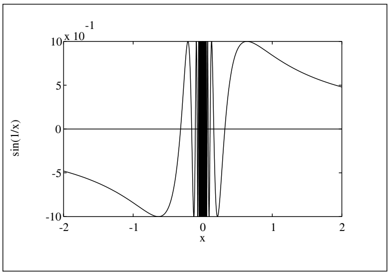

Fig. 1.2. Graph of $\sin(1/x)$

Example 1.4. The function $f(x)$, defined by

$$f(x) = \begin{cases} e^{-1/x^2} sin(1/x), & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases}$$

is not analytic since $x = 0$ is a point of accumulation for the zeros of f. However, it is a C^∞ function.

In the multivariable case, one can analogously consider an open domain $D \subset I\!R^n,\ n \in I\!N$ and the following Definition.

Definition 1.5. *A function $f : D \to I\!R$ is said analytic in D if it coincides with its Taylor expansion in the neighborhood of every point $x_0 \in D$.*

The generalization of Proposition 1.2 becomes

Proposition 1.6. *Let $D \subseteq I\!R$ be a convex, open domain and let $f : D \to I\!R$ be an analytic function on D, then either*

(i) $f \equiv 0$ on D, or
(ii) the set of zeros of f in D has an empty interior.

Proof. Given any point $x_1 \in D$, take a point x_0, if any exists, in the interior of the set of zeros of f in D. The straight line L that passes through x_1 and x_0 contains an interval whose points are zeros of f and, henceforth, of the restriction of f to L. Since the restriction of f to L is analytic, it is zero by Proposition 1.2, and hence $f(x_1) = 0$. ∎

Example 1.7. Let $f : I\!R^2 \to I\!R$, be defined by $f(x_1, x_2) = x_1 - 1$. It is easily seen that f is analytic. For any point $P \in I\!R^2$ which belongs to the set $\mathcal{Z}$ of zeros of f, there does not exist any $I\!R^2$-neighborhood included in $\mathcal{Z}$ (see Figure 1.7).

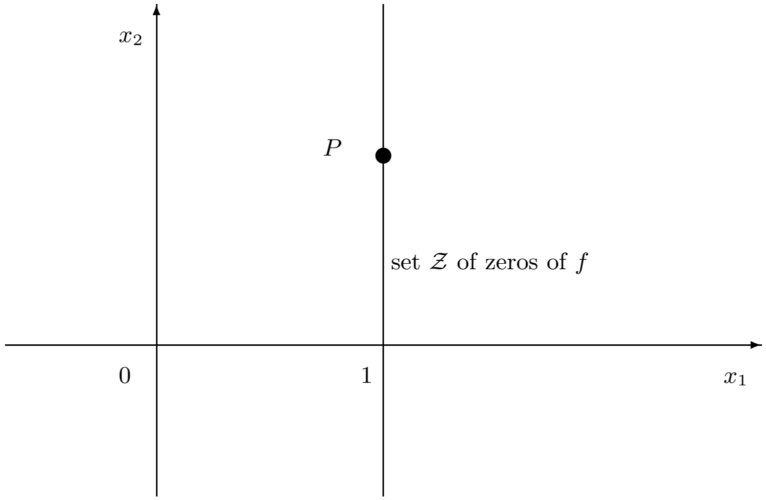

Fig. 1.3. Zeros of $f(x_1, x_2) = x_1 - 1$

By Proposition 1.6, nonzero analytic functions defined on $I\!R^n$ are different from 0 at the points of an open (since the set of zeros is obviously closed) and dense subset of $I\!R^n$, or, in other terms, they are generically different from 0. Then, it makes sense to define the *generic rank* of a matrix whose entries are analytic functions as the dimension of the maximum square submatrix having a nonzero determinant. As the determinant is an analytic function, the generic rank coincides with the rank of the matrix at the points of an open, dense subset of $I\!R^n$. Moreover, the generic rank is greater than or equal to the rank at any point of $I\!R^n$. Recalling what we said about the rank of the vector field $g(x)$ appearing in (1.1) and our interest in it being generically maximal, it should be clear, now, that we are motivated to assume that the functions f, g, and h in (1.1) are analytic.
If a function $f(x)$ is analytic, in general, the same is not true for its multiplicative inverse $1/f(x)$. However, in a suitable algebraic framework, we can give a notion of the multiplicative inverse of a nonzero analytic function. To explain this, let us first note that, with the usual notions of sum, denoted by $+$, and of product, denoted by $\cdot$, the set of analytic functions from $I\!R^r$ to $I\!R$ forms a ring, denoted by $\mathcal{A}_r$. An important ring theoretical property of $\mathcal{A}_r$ is

that it does not contain zero divisors.[1]
Note that C^∞ functions too form a ring, but in that ring there are zero divisors, for instance, the nonzero elements

$$f_1(x) = \begin{cases} e^{-1/x^2}, & \text{if} x < 0 \\ 0, & \text{if} x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if} x \leq 0 \\ e^{-1/x^2}, & \text{if} x > 0 \end{cases}$$

whose product is identically zero. Rings that do not have zero divisors are called *integral domains* and possess several nice properties (see, e.g., [5] for classification of rings and for generalities about the following construction). Here, we are interested in the fact that an integral domain can be naturally embedded in a larger algebraic object, called its quotient field, to provide a notion of multiplicative inverse to every nonzero element. The construction of the quotient field of $\mathcal{A}_r$, called $\mathcal{K}_r$, is quite general. The elements of $\mathcal{K}_r$ are pairs (f, g) of elements of $\mathcal{A}_r$ such that $g \neq 0$, modulo the equivalence relation $\approx_R$ defined by $(f, g) \approx_R (f', g')$ if and only if $fg' = gf'$. Choosing a representative in the equivalence class, an element of $\mathcal{K}_r$ will be written as f/g. Using representatives, the sum and product, still denoted by $+$ and $\cdot$, of two elements f_1/f_2 and g_1/g_2 of $\mathcal{K}_r$ can be defined as follows:

$$(f_1, g_1) + (f_2, g_2) := (f_1 \cdot g_2 + f_2 \cdot g_1, g_1 \cdot g_2)$$
$$(f_1, g_1) \cdot (f_2, g_2) := (f_1 \cdot f_2, g_1 \cdot g_2)$$

Note that the above definitions are well posed in $\mathcal{K}_r$, since $g_1 \cdot g_2 \neq 0$ and the result of the sum and product does not depend on the chosen representatives. With the above operation, $\mathcal{K}_r$ is a field. The ring $\mathcal{A}_r$ can be identified with a subring of $\mathcal{K}_r$ mapping any element $f \in \mathcal{A}_r$ to $f/1 \in \mathcal{K}_r$. Given $f \in \mathcal{A}_r$, $f \neq 0$, its inverse in $f \in \mathcal{K}_r$ is $1/f$. In the following, we will write f for the element $f/1$ and fg for the product $f \cdot g$ in $\mathcal{K}_r$.
The elements of the quotient field $\mathcal{K}_r$ of the ring of analytic functions are called *meromorphic functions* .
Any rational function is a meromorphic function; another typical example is $\tan x = \sin x / \cos x$. If we look to meromorphic functions as functions of a real n-dimensional variable, we see that their domains of definition are open and dense subsets of $\mathbb{R}^n$. At the same time, their sets of zeros have empty interiors.

1.2 Control Systems

As mentioned in Section 1, we can now state precisely that the class of dynamic systems we are going to deal with basically consists of objects defined by a

[1] In a ring A, a nonzero element x is a *zero divisor* if there exists some other nonzero element y such that $x \cdot y = 0$.

set of first-order differential equations of the form

$$\Sigma = \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases} \tag{1.4}$$

where the independent variable $t \in I\!R$ denotes time; the state $x(\cdot) \in I\!R^n$; the input $u(\cdot) \in I\!R^m$; the output $y(\cdot) \in I\!R^p$; and the entries of f, g, h are meromorphic functions.

In addition, we ask that the following assumption is satisfied.

Assumption 1.8 *Given a system Σ of the form (1.4), the matrix $g(x)$ is such that rank $g = m$.*

Dynamic systems of the above kind are usually called *control systems* or *nonlinear control systems* if one wants to stress the fact that f and h are generally nonlinear and g is not constant. As no confusion is possible, we will usually drop the variable t from equations (1.4) and others of the same kind. We will refer to the representation (1.4) as the *state-space representation* or *internal representation* of a control system. External representations, on which the state variable does not appear, will be discussed in Chapter 2, together with their relation to internal representations.
A remarkable property of the system of differential equations (1.4) is that it is affine in the input variable u. In principle, this is a nontrivial restriction of the class of dynamic systems we want to consider, since more general models could be obtained by substituting (1.4) with

$$\Sigma = \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{1.5}$$

where the variables $x \in I\!R^n$, $u \in I\!R^m$, and $y \in I\!R^p$ have the usual meaning and the entries of f, g, and h are meromorphic functions. However, affine control systems, that is dynamic systems defined by (1.4), are sufficiently general for modelling purposes.
In addition, the class of nonlinear systems affine in the control is closed with respect to basic system theoretical operations, such as series and parallel composition, and, what is crucial, state feedback of the form $u(t) = \alpha(x(t)) + \beta(x(t))v(t)$ where the entries of $\alpha(x)$ and $\beta(x)$ are meromorphic functions. This is not true for systems defined by equations of the form (1.5) because, for instance, for $f(x, u) = \sin\ u$, the choice of a feedback $u = 1/x$ would give rise to a representation whose elements are not meromorphic functions.
Although in some situations we will need to consider dynamic systems of a more general kind and more exotic feedbacks than those described above, we will in general deal with objects that have an affine representation in u.

1.3 Linear Algebraic Setting

Assuming that a nonlinear control system Σ of the form (1.1) is given, the objective of this section is to construct an algebraic setting for defining and studying the system theoretical properties of Σ. Our approach is built up by introducing the notion of differential form in an abstract and formal way. This choice is motivated by simplicity and by the fact that in the rest of the book we will be interested only in the abstract algebraic and formal properties of differential forms. Other less abstract treatments of the same topics, like those developed in [83, 101, 126, 157] agree at a formal level with the one described here. Our approach has contact points with that of [19] and, to avoid technicalities, the reader is referred to [19] for proofs and technical constructions not found here.

To begin with, denoting by n and m, respectively, the dimensions of the state space and of the input space of the system Σ, let us consider the infinite set of real indeterminates

$$\mathcal{C} = \{x_i, i = 1, \ldots, n; u_j^{(k)}, j = 1, \ldots, m, k \geq 0\}$$

For any positive integer r, we use the first r elements of $\mathcal{C}$ to denote the coordinates of a point in $\mathbb{R}^r$. Hence, a function from $\mathbb{R}^r$ to $\mathbb{R}$, in particular an element of $\mathcal{K}_r$, will be written as a function in the first r indeterminates of $\mathcal{C}$.

The usual partial derivative operators $\partial/\partial x_i$ and $\partial/\partial u_j^{(k)}$ act naturally on the field $\mathcal{K}_r$ of all meromorphic functions from $\mathbb{R}^r$ to $\mathbb{R}$, which, for that reason, is said to be endowed with a *differential field structure*. Differential fields and their properties will not be explicitly employed as tools in this book. However, the reader can find a brief introduction to them, together with an essential description of the point of view one can develop on control theory starting from differential algebra, in [26], as well as in the third part of [23]. More information can be found in [98, 102, 140]. Here, it is sufficient for our aims to remark that, letting $\mathcal{K}$ denote the set theoretical union $\bigcup_r \mathcal{K}_r$, $\mathcal{K}$ has an obvious field structure, and, moreover, it can be endowed with a differential structure determined by the system Σ. Because any element of $\mathcal{K}$ is a meromorphic function depending on a finite subset of indeterminates of $\mathcal{C}$ and, consequently, can be in general denoted by $F(\{x_i, u_j^{(k)}\})$, we can define a derivative operator δ, acting on $\mathcal{K}$, as follows :

$$\delta x_i = f_i(x) + g_i(x)u^{(0)} \text{ for all } \quad i = 1, \ldots, n$$

$$\delta u_j^{(k)} = u_j^{(k+1)} \text{ for } k \geq 0 \text{ and for all } \quad j = 1, \ldots, m$$

$$\delta F(\{x_i, u_j^{(k)}\}) = \sum_{i=1}^{n} (\partial F/\partial x_i)\delta x_i + \sum_{j=1,\ldots,m;k\geq 0} (\partial F/\partial u_j^{(k)})\delta u_j^{(k)}$$

The resulting differential field is the starting point for a number of constructions that will be used in characterizing the system theoretical properties of Σ.

1.3.1 One-forms

We consider now the infinite set of symbols

$$\mathrm{d}\mathcal{C} = \{\mathrm{d}x_i, i = 1, \dots, n; \mathrm{d}u_j^{(k)}, j = 1, \dots, m, k \geq 0\} \tag{1.6}$$

and we denote by $\mathcal{E}$ the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$, namely

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}} \mathrm{d}\mathcal{C}. \tag{1.7}$$

Any element in $\mathcal{E}$ is a vector of the form

$$v = \sum_{i=1}^{n} F_i \mathrm{d}x_i + \sum_{j=1,\dots,m;k\geq 0} F_{jk} \mathrm{d}u_j^{(k)}$$

where only a finite number of coefficients F_{jk} are nonzero elements of $\mathcal{K}$. We can define now an operator from $\mathcal{K}$ to $\mathcal{E}$, which by abuse of notation will be denoted by d, in the following way:

$$\mathrm{d}F(\{x_i, u_j^{(k)}\}) = \sum_{i=1}^{n} (\partial F/\partial x_i)\mathrm{d}x_i + \sum_{j=1,\dots,m;k\geq 0} (\partial F/\partial u_j^{(k)})\mathrm{d}u_j^{(k)}$$

The elements of $\mathcal{E}$ will be called *one-forms* and we will say that $v \in \mathcal{E}$ is an *exact* one-form, or that it is *integrable*, if $v = \mathrm{d}F$ for some $F \in \mathcal{K}$. We will usually refer to $\mathrm{d}F$ as to the *differential* of F.

Example 1.9. Let $F = \sin(x_1 x_2) \in \mathcal{K}$. Then,

$$\mathrm{d}F = \cos(x_1 x_2)\, [x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2] \in \mathcal{E}$$

The vector space of one-forms $\mathcal{E}$ can be endowed with a differential structure by defining a derivative operator Δ, in terms of the derivative operator δ and of the differential operator d acting on $\mathcal{K}$, as follows:

$$\begin{aligned} \Delta v &= \Delta(\textstyle\sum_{i=1}^{n} F_i \mathrm{d}x_i + \sum_{j=1,\dots,m;k\geq 0} F_{jk} \mathrm{d}u_j^{(k)}) \\ &= \textstyle\sum_{i=1}^{n} (\delta F_i \mathrm{d}x_i + F_i \mathrm{d}(\delta x_i)) + \sum_{j=1,\dots,m;k\geq 0} (\delta F_{jk} \mathrm{d}u_j^{(k)} + F_{jk} \mathrm{d}(\delta u_j^{(k)})) \end{aligned}$$

1.3.2 Two-forms

We consider now the infinite set of symbols

$$\begin{aligned} \wedge \mathrm{d}\mathcal{C} = \{&\mathrm{d}x_i \wedge \mathrm{d}x_{i'}; \mathrm{d}u_j^{(k)} \wedge \mathrm{d}u_{j'}^{(k')}; \mathrm{d}x_i \wedge \mathrm{d}u_j^{(k)}; \mathrm{d}u_j^{(k)} \wedge \mathrm{d}x_i, \text{ for i} = 1, \dots, n \\ & i' = 1, \dots, n; j = 1, \dots, m; j' = 1, \dots, m; k \geq 0; k' \geq 0\} \end{aligned}$$

and the vector space, that we denote by $\wedge\mathcal{E}$, spanned over $\mathcal{K}$ by the elements of $\wedge \mathrm{d}\mathcal{C}$. In $\wedge\mathcal{E}$, we consider the equivalence relation R spanned by the equalities

$$\mathrm{d}\alpha \wedge \mathrm{d}\beta = -\mathrm{d}\beta \wedge \mathrm{d}\alpha \tag{1.8}$$

Note that the above relation implies that $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$ for every i and $\mathrm{d}u_j^{(k)} \wedge \mathrm{d}u_j^{(k)} = 0$ for every j and k.
The vector space $\wedge\mathcal{E}$ mod R will be denoted in the following by $\mathcal{E}^{(2)}$. The elements of $\mathcal{E}^{(2)}$ are called *two-forms*.
We can define now an operator from $\mathcal{E}$ to $\mathcal{E}^{(2)}$, that by abuse of notation will again be denoted by d, in the following way :

$$\begin{aligned}
\mathrm{d}v = \mathrm{d}(\sum_{i=1}^{n} F_i \mathrm{d}x_i + & \sum_{\substack{j=1,\dots,m\\ k\geq 0}} F_{jk}\mathrm{d}u_j^{(k)}) \\
= & \sum_{\substack{i=1,\dots,n\\ i'=1,\dots,n}} (\delta F_i/\delta x_{i'})\mathrm{d}x_{i'} \wedge \mathrm{d}x_i + \sum_{\substack{i=1,\dots,n\\ j=1,\dots,m\\ k\geq 0}} (\delta F_i/\delta u_j^{(k)})\mathrm{d}u_j^{(k)} \wedge \mathrm{d}x_i \\
& + \sum_{\substack{i=1,\dots,n\\ j=1,\dots,m\\ k\geq 0}} (\delta F_{jk}/\delta x_i)\mathrm{d}x_i \wedge \mathrm{d}u_j^{(k)} \\
& + \sum_{\substack{j=1,\dots,m\\ j'=1,\dots,m\\ k\geq 0}} (\delta F_{jk}/\delta u_{j'}^{(k)})\mathrm{d}u_{j'}^{(k)} \wedge \mathrm{d}u_j^{(k)}) \bmod R
\end{aligned}$$

A canonical representative of $\mathrm{d}v$ is given by

$$\begin{aligned}
\mathrm{d}v = & \sum_{i>i'} (\delta F_i/\delta x_{i'} - \delta F_{i'}/\delta x_i)\mathrm{d}x_{i'} \wedge \mathrm{d}x_i \\
& + \sum_{i=1,\dots,n;j=1,\dots,m;k\geq 0} (\delta F_i/\delta u_j^{(k)} - \delta F_{jk}/\delta x_i)\mathrm{d}u_j^{(k)} \wedge \mathrm{d}x_i \\
& + \sum_{j>j';k\geq 0;k'\geq 0} (\delta F_{jk}/\delta u_{j'}^{(k')} - \delta F_{j'k'}/\delta u_j^{(k)})\mathrm{d}u_{j'}^{(k')} \wedge du_j^{(k)}
\end{aligned}$$

Example 1.10. Let $v = \mathrm{d}x_1 - (x_1/x_2)\mathrm{d}x_2$, then

$$\mathrm{d}v = 0 - (1/x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

1.3.3 s-forms

We need now to consider a more general construction of the same kind as that giving rise to $\mathcal{E}^{(2)}$. To this aim, let us consider, for any integer s, the infinite set of symbols

$$\wedge^s \mathrm{d}\mathcal{C} = \{\mathrm{d}\xi_0 \wedge \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_s; \xi_i \in \mathcal{C}; i = 0, \ldots, s\}$$

and the vector space that we denote by $\wedge^s \mathcal{E}$ spanned over $\mathcal{K}$ by the elements of $\wedge^s \mathrm{d}\mathcal{C}$. In $\wedge^s \mathcal{E}$, we consider the equivalence relation R spanned by the equalities

$$\mathrm{d}\xi_{i_0} \wedge \mathrm{d}\xi_{i_1} \wedge \ldots \wedge \mathrm{d}\xi_{i_s} = (-1)^\sigma \mathrm{d}\xi_{j_0} \wedge \mathrm{d}\xi_{j_1} \wedge \ldots \wedge \mathrm{d}\xi_{j_s}$$

where σ is the signature of the permutation $\begin{pmatrix} i_0 & \cdots & i_s \\ j_0 & \cdots & j_s \end{pmatrix}$. The vector space $\wedge^s \mathcal{E}$ mod R will be denoted in the following by $\mathcal{E}^{(s+1)}$, its elements are called $(s+1)$-*forms.*

Note that the above relation implies that $\mathrm{d}\xi_{i_0} \wedge \mathrm{d}\xi_{i_1} \wedge \ldots \wedge \mathrm{d}\xi_{i_s} = 0$ if $\mathrm{d}\xi_{i_j} = \mathrm{d}\xi_{i_k}$ for some index j and k.

By the constructions described above, we obtain a set of vector spaces $\mathcal{E}$, $\mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(s)}$ that are related to the system Σ. These algebraic objects will be the basic tools for analyzing the system properties of Σ in Chapters 3 to 6 and for solving several design problems in the second part of the book.

1.3.4 Exterior Product

The *exterior product* or *wedge product* of a p-form ω_1 and a q-form ω_2, denoted by $\omega_1 \wedge \omega_2$, can now be defined as the $(p+q)$-form whose representative, if ω_1 is represented as

$$\omega_1 = \sum_{i=1,\ldots,k} F_i \xi_i^{(p)}$$

with $\xi_i^{(p)} \in \wedge^{p-1} \mathrm{d}\mathcal{C}$ and ω_2 is represented as

$$\omega_2 = \sum_{j=1,\ldots,h} G_j \xi_j^{(q)}$$

with $\xi_j^{(q)} \in \wedge^{q-1} \mathrm{d}\mathcal{C}$, is given by

$$\omega_1 \wedge \omega_2 = \sum_{i=1,\ldots,k; j=1,\ldots,h} F_i G_j \xi_i^{(p)} \wedge \xi_j^{(q)}$$

It can easily be verified that the exterior product is associative. Moreover, it induces a map $\wedge : \mathcal{E}^{(p)} \times \mathcal{E}^{(q)} \to \mathcal{E}^{(p+q)}$ given, quite obviously, by $\wedge(\omega_1, \omega_2) = \omega_1 \wedge \omega_2$.

1.4 Frobenius Theorem

In this section, we investigate the problem of checking the exactness of a given one-form $v \in \mathcal{E}$, namely, if $v = \mathrm{d}\varphi$ for some $\varphi \in \mathcal{K}$. To begin with, let us first state some elementary facts.

Definition 1.11. *A one-form $v \in \mathcal{E}$ is closed if $\mathrm{d}v = 0$.*

Proposition 1.12. *Any exact one-form is closed.*

Proof. Consider a function $\varphi \in \mathcal{K}$ of n variables, say $\xi_1, \cdots, \xi_n$ with $\xi_i \in \mathcal{C}$. Then, $\mathrm{d}\varphi = \sum_{i=1}^{i=n} \frac{\partial \varphi}{\partial \xi_i} \mathrm{d}\xi_i$ and

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\varphi) &= \textstyle\sum_{i,j} \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_i} \mathrm{d}\xi_j \wedge \mathrm{d}\xi_i \\
&= \textstyle\sum_{i \geq j} \left(\frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_i} \right) \mathrm{d}\xi_i \wedge \mathrm{d}\xi_j \\
&= 0
\end{aligned}$$

∎

Poincaré's Lemma (see [19] for a proof) establishes that the converse of Proposition 1.12 is true only locally.

Lemma 1.13. Poincaré's Lemma *Let v be a closed one-form in $\mathcal{E}$. Then, there exists $\varphi \in \mathcal{K}$ such that locally $v = \mathrm{d}\varphi$.*

Example 1.14. A typical example of a closed form that is not exact is the following. In $\mathbb{R}^2$, consider the closed one-form $\omega = \frac{x_2}{x_1^2 + x_2^2} \mathrm{d}x_1 - \frac{x_1}{x_1^2 + x_2^2} \mathrm{d}x_2$. Locally, around any point (x_1, x_2) such that $x_2 \neq 0$, $\omega = \mathrm{d}\left[\arctan(x_1/x_2)\right]$, and around any point (x_1, x_2) such that $x_2 = 0$ and $x_1 \neq 0$, $\omega = \mathrm{d}\left[\arctan(-x_2/x_1)\right]$. But there is no function φ such that $\omega = \mathrm{d}\varphi$ globally.

A requirement weaker than exactness for a one-form v is that of being colinear to an exact form, *i.e.* there exist λ and φ in $\mathcal{K}$ such that $\lambda v = \mathrm{d}\varphi$ or, equivalently, that $\mathrm{span}_{\mathcal{K}}\{v\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi\}$. A function λ such that $\lambda v = \mathrm{d}\varphi$ is called an *integrating factor*. The characterization of this property is a special case of the Frobenius Theorem that will be stated later.

Theorem 1.15. *Given $v \in \mathcal{E}$, there exists a function φ such that $\mathrm{span}_{\mathcal{K}}\{v\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi\}$ if and only if*

$$\mathrm{d}v \wedge v = 0$$

Proof. Necessity: Since $\mathrm{d}\varphi \in \mathrm{span}_{\mathcal{K}}\{v\}$, there exists a nonzero function α such that $\alpha v = \mathrm{d}\varphi$. Hence, αv is exact, that is, $\mathrm{d}(\alpha v) = 0$. From

$$0 = \mathrm{d}(\alpha v) = \mathrm{d}\alpha \wedge v + \alpha \mathrm{d}v$$

it follows

$$0 = \mathrm{d}(\alpha v) \wedge v \ = 0 + \alpha \mathrm{d}v \wedge v$$

which yields the desired result.
Sufficiency: The proof of sufficiency is not elementary and for that the reader is referred to [19, 46]. ∎

To generalize the result of Theorem 1.15, let us introduce the following definition.

Definition 1.16. *A subspace $V \subset \mathcal{E}$ is closed, or integrable, if V has a basis which consists only of closed forms.*

Theorem 1.17. Frobenius Theorem *Let $V = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. V is closed if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0, \text{ for any } i = 1, \ldots, r$$

Example 1.18.

- The one-form $\omega = x_1 \mathrm{d}x_1 + x_2 \mathrm{d}x_2$ is closed according to Definition 1.11. In fact, $\omega = \frac{1}{2}\mathrm{d}(x_1^2 + x_2^2)$.
- The one-form $\omega = \mathrm{d}x_1 + x_1 \mathrm{d}x_2$ is not closed since $\mathrm{d}\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \neq 0$. However, the vector space $\mathrm{span}_{\mathcal{K}}\{\omega\}$ is integrable since $\mathrm{d}\omega \wedge \omega = 0$ and one may choose the integrating factor $\alpha = 1/x_1$.

The version of the Frobenius Theorem stated above is dual of the version commonly presented in the literature. The reason for this choice is that it fits more naturally with our formalism. The reader can find a proof of the Theorem in [19], or also, in dual terms, in [46, 118].
The necessary and sufficient condition of the Frobenius Theorem is verified, in particular, if V has dimension $n-1$ and its generators depend on n indeterminates. Let us state and prove explicitly that this results.

Proposition 1.19. *Let $V = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_{n-1}\}$ be a $(n-1)$-dimensional subspace of $\mathcal{E}$ and assume that its generators ω_i, for $i = 1, ..., n-1$, depend on n indeterminates, say $x_1, ..., x_n$. Then, V is closed (integrable).*

Proof. Denote by $X(x)$, where $x = (x_1, ..., x_n)^T$, the annihilator of V, so that

$$\omega_i(x) \cdot X(x) \equiv 0$$

for any $i = 1, ..., n-1$. Without loss of generality, assume that

$$X(x) = \begin{bmatrix} 1 \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

and consider the differential equation

$$\dot{x}(t) = X(x(t))$$

with initial condition $x(0) = x_0$. The solution $\Phi_t^X(x_0)$ of such an equation can be written in the form

$$\Phi_t^X(x_0) = \begin{bmatrix} t + x_{10} \\ \phi_2(t, x_0) \\ \vdots \\ \phi_n(t, x_0) \end{bmatrix}$$

The variable t can be formally eliminated using the equality $t = x_1 - x_{10}$, so that

$$x_i = \phi_i(x_1 - x_{10}, x_0)$$

for $i = 2, ..., n$. Defining

$$\begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_n(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 + \phi_2(x_1 - x_{10}, x_0) \\ \vdots \\ -x_n + \phi_n(x_1 - x_{10}, x_0) \end{bmatrix}$$

one easily checks that $\dot{p}_i = 0$ or, equivalently, that $\mathrm{d}p_i(x) \perp X(x)$ for $i = 2, ..., n$. Since $\operatorname{rank} \frac{\partial p(x)}{\partial x} = n$, $p(x)$ defines a change of coordinates and $V = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}p_2, \ldots, \mathrm{d}p_n\}$. ∎

1.5 Examples

The algebraic formalism of one-forms is devised mainly for facilitating computations involving gradients and Jacobian matrices.

Note, in particular, that the differential $\mathrm{d}y$ of the output of the control system (1.1) is a vector in $\mathcal{E}$:

$$\mathrm{d}y = \frac{\partial h}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial h}{\partial x_n}\mathrm{d}x_n$$

The differential of the kth time derivative of y is in $\mathcal{E}$ as well.

Example 1.20. The one-form $\omega_1 = \mathrm{d}x_1 \in \mathcal{E}$ may be thought of, with respect to the basis $\mathrm{d}\mathcal{C}$, as the *row vector* $\begin{bmatrix} 1 \; 0 \; 0 \cdots \end{bmatrix}$. This row vector is the Jacobian $\begin{bmatrix} \frac{\partial f}{\partial x_1} \; \frac{\partial f}{\partial x_2} \cdots \end{bmatrix}$ of the function $f(x) = x_1$, which belongs to $\mathcal{K}$, and it is obviously exact.

Example 1.21. Let us consider the one-form $\omega_2 = x_3 \mathrm{d}x_1 \in \mathcal{E}$, which may be identified with the row vector $\omega_2 = \begin{bmatrix} x_3 & 0 & 0 & \cdots \end{bmatrix}$. The exactness or integrability of ω_2 is equivalent to the existence of a function $\psi \in \mathcal{E}$, such that $\omega_2 = \begin{bmatrix} \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} & \frac{\partial \psi}{\partial x_3} & \cdots \end{bmatrix}$. If such a ψ does exist, then necessarily it solves the system

$$\begin{cases} \dfrac{\partial \psi}{\partial x_1} = x_3 \\ \dfrac{\partial \psi}{\partial x_3} = 0 \end{cases}$$

Using second-order derivatives, one concludes that there is no solution since

$$\frac{\partial}{\partial x_3}\frac{\partial \psi}{\partial x_1} = 1 \neq \frac{\partial}{\partial x_1}\frac{\partial \psi}{\partial x_3} = 0$$

Examples of this kind motivate a formal study of second-order derivatives and give rise to the notion of two-forms, that generalize somehow Hessian matrices, consisting of second-order partial derivatives.

Example 1.22. Let $v = (1/x_2)\mathrm{d}x_1 - (x_1/x_2^2)\mathrm{d}x_2$. To check the closure (or local exactness) of v, one may proceed as above and compute $\frac{\partial}{\partial x_2}\frac{1}{x_2}$ and $\frac{\partial}{\partial x_1}\left(\frac{-x_1}{x_2^2}\right)$. The two-form $\mathrm{d}v$ embodies these computations:

$$\mathrm{d}v = \frac{\partial}{\partial x_2}\left(\frac{1}{x_2}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_1 + \frac{\partial}{\partial x_1}\left(\frac{-x_1}{x_2^2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{1.9}$$

$$= -(1/x_2^2)\mathrm{d}x_2 \wedge \mathrm{d}x_1 - (1/x_2^2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{1.10}$$

Now, the closure of v results from the fact that $\frac{\partial}{\partial x_2}\frac{1}{x_2} = \frac{\partial}{\partial x_1}\frac{-x_1}{x_2^2}$, or, since in $\mathcal{E}^{(2)}$, $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\mathrm{d}x_2 \wedge \mathrm{d}x_1$, then $\mathrm{d}v = 0$. In fact, $v = \mathrm{d}(x_1/x_2)$.

In the case of Example 1.20 and of Example 1.21, respectively,

$$\mathrm{d}\omega_1 = \mathrm{d}(\mathrm{d}x_1) = 0$$

$$\mathrm{d}\omega_2 = \mathrm{d}(x_3\mathrm{d}x_1) = \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3\mathrm{d}(\mathrm{d}x_1) = \mathrm{d}x_3 \wedge \mathrm{d}x_1$$

where $\mathrm{d}\omega_2 \neq 0$ displays the fact that ω_2 is not exact. In fact, as a linear combination of the symbols $\mathrm{d}x_1 \wedge \mathrm{d}x_3$ and $\mathrm{d}x_3 \wedge \mathrm{d}x_1$, $\mathrm{d}\omega_2$ reads as

$$\begin{array}{ccccc} \mathrm{d}\omega_2 = & 0 & \mathrm{d}x_1 \wedge \mathrm{d}x_3 + & 1 & \mathrm{d}x_3 \wedge \mathrm{d}x_1 \\ & \downarrow & & \downarrow & \\ & \text{candidate for} & & \text{candidate for} & \\ & \dfrac{\partial}{\partial x_1}\dfrac{\partial \psi}{\partial x_3} & & \dfrac{\partial}{\partial x_3}\dfrac{\partial \psi}{\partial x_1} & \end{array}$$

Since $\frac{\partial}{\partial x_1}\frac{\partial \psi}{\partial x_3} = \frac{\partial}{\partial x_3}\frac{\partial \psi}{\partial x_1}$ for any ψ and $\mathrm{d}x_1 \wedge \mathrm{d}x_3 = -\mathrm{d}x_3 \wedge \mathrm{d}x_1$, $\mathrm{d}\omega_2 \neq 0$ formalizes the fact that there is no function $\psi \in \mathcal{K}$ such that $\omega_2 = \mathrm{d}\psi$.

Problems

1.1. Prove that invertibility is a generic property for real square matrices.

1.2. Prove that square, symmetric, real matrices are not generically positive definite.

1.3. The dynamic system Σ defined by (1.1) is linear if the vector field $g(x)$ is constant, namely, $g(x) = B$ for a suitable matrix B, and the functions $f(x)$ and $h(x)$ are linear, namely $f(x) = Ax$ and $h(x) = Cx$ for suitable matrices A and C. For a linear system Σ, controllability, namely, the possibility to drive the state to zero in finite time by applying a suitable input $u(t)$, is equivalent to the condition

$$rank[BABA^2B...A^{n-1}B] = n$$

n being the dimension of A. Prove that controllability is a generic property of linear systems.

1.4. Prove that the function $f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x < 0 \\ 0, & \text{if} x \geq 0 \end{cases}$ is not analytic at $x = 0$.

1.5. Integration of one-forms
Check if the following one-forms are exact and in case of a positive answer, find a function F whose differential coincides with them.
(a) $(1 + cos(x + y))dx + cos(x + y)dy$
(b) $\dfrac{x + 2y}{x^3 y}dx + \dfrac{1}{xy^2}dy$
(c) $\dfrac{x}{x^2 + y^2}dx + \dfrac{y}{x^2 + y^2}dy$
(d) $-\dfrac{y}{x^2 + y^2}dx + \dfrac{x}{x^2 + y^2}dy$
(e) $\dfrac{x}{x^2 + y^2 + z^2}dx + \dfrac{y}{x^2 + y^2 + z^2}dy + \dfrac{z}{x^2 + y^2 + z^2}dz$
(f) $-\dfrac{y + z}{(x - y - z)^2}dx + \dfrac{x}{(x - y - z)^2}dy + \dfrac{x}{(x - y - z)^2}dz$

1.6. Check if the one-form $\omega = (-x^3 cos(y))dx + (xsin(y))dy$ is closed. If ω is not a closed one-form, check if an integrating factor exists and in case of a positive answer, compute it.

1.7. Exterior differentiation
Compute the differential of the following differential forms:
(a) $sin(x + y)dx + (x^2 + 2y)dy + zdz$
(b) $cos(z)dx$
(c) $(x^2)dx \bigwedge dy$
(d) $(e^x)dx \bigwedge dy + x\ dy \bigwedge dz$
(e) $dx \bigwedge dy \bigwedge dz$

1.8. Prove that $\mathrm{d}\xi_{i_0} \wedge \mathrm{d}\xi_{i_1} \wedge \ldots \wedge \mathrm{d}\xi_{i_s} = 0$ if $\mathrm{d}\xi_{i_j} = \mathrm{d}\xi_{i_k}$ for some index j and k.

1.9. Exterior product
Compute the exterior product between k-forms.
(a) $dx \bigwedge (sin(y)dy \bigwedge (x\ dx + (y^2)dy)$
(b) $(cos(xy)dx + (y^3)dy) \bigwedge (z\ dx + y\ dz)$
(c) $(2x\ dx + (x+y)^2 dy + (1-z)dz) \bigwedge (ydx - xdz)$
(d) $(e^x)dx \bigwedge dy + x\ dy \bigwedge dz$
(e) $(dx \bigwedge dy) \bigwedge (cos(x+y)dy \bigwedge dz)$

2

Modeling

Dynamic systems may be described in several ways. Physics often yields descriptions in terms of high order differential equations that involve input and output variables. Starting from this situation, a typical control theoretical problem is that of restating such input-output descriptions in terms of coupled, first-order differential equations, introducing new instrumental internal variables or states. For linear systems, it is possible to switch easily from input-output, or external, representation to state-space, or internal, representation, using the Laplace transform to change the domain of the representation. Although such a tool for symbolic computation is not available for nonlinear systems, we show, in this chapter, that internal, state-space representation can be derived from input-output descriptions (their construction will be described as the realization problem) and conversely, external, input-output descriptions can be derived from state-space descriptions (their construction will be described as the state elimination problem) in a nonlinear context, too.
To develop the tools required for dealing with this kind of problem, we will start by considering first the state elimination problem and, then, we will tackle the more relevant problem of constructing state-space representations from input/output relations.

2.1 State Elimination

Given the internal, or state-space, description of a system Σ, it is possible, in a sense to be made precise, to construct a representation of the relationship between input and output that it defines in a form that does not involve state variables. Although the validity of such a representation is only local, it nevertheless turns out to be useful for understanding the system behavior and, more important, its construction helps in clarifying the inverse problem of defining state variables and state equations from an input/output relation. To describe the situation, we can consider, without additional difficulties, internal representations more general than (**??**), *i.e.*, representations of the form

$$\begin{cases} \dot{x} = f(x, u, \ldots, u^{(s)}) \\ y = h(x, u, \ldots, u^{(s)}) \end{cases} \tag{2.1}$$

where, as usual, $x \in I\!R^n$, $u \in I\!R^m$, and $y \in I\!R^p$, and the entries of f and h, which depend also on a finite number of time derivatives of the input, are analytic functions.
So, given a system Σ of the form (2.1), the problem is to find, if possible, a set of input-output differential equations of the form

$$F_i(y, \dot{y}, \ldots, y^{(k)}, u, \dot{u}, \ldots, u^{(\gamma)}) = 0 \quad , \quad i = 1, \ldots, p \tag{2.2}$$

which admits as solution any pair $(y(t), u(t))$ such that $(y(t), u(t), x(t))$ is a solution, for some $x(t)$, of (2.1). Such a set of differential equations, if any exists, will be called an external, or input-output, representation of the system Σ described by (2.1).

Theorem 2.1. *Given a system Σ of the form (2.1), where the entries of f and h are analytic functions, there exist an integer γ and an open dense subset V of $I\!R^{n+m\gamma}$ such that, in the neighborhood of any point of V, there exists an input-output representation of the system of the form (2.2).*

Proof. The first step in constructing an input-output representation consists of applying a suitable change of coordinates. To this aim, let us denote by s_1 the minimum nonnegative integer such that

$$\text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)})}{\partial x} = \text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1)})}{\partial x}$$

If $\partial h_1/\partial x \equiv 0$ we define $s_1 = 0$. Analogously for $1 < j \leq p$, let us denote by s_j the minimum integer such that

$$\text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)}; \ldots; h_j, \ldots, h_j^{(s_j-1)})}{\partial x}$$

$$= \text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)}; \ldots; h_j, \ldots, h_j^{(s_j)})}{\partial x}$$

If

$$\text{rank}\frac{\partial(h_1, \ldots, h_{j-1}^{(s_{j-1}-1)})}{\partial x} = \text{rank}\frac{\partial(h_1, \ldots, h_{j-1}^{(s_{j-1}-1)}, h_j)}{\partial x}$$

we define $s_j = 0$. Write $K = s_1 + \ldots + s_p$. The vector

$$S = (h_1, \ldots, h_1^{s_1-1}, \ldots, h_p, \ldots, h_p^{s_p-1})$$

where h_j does not appear if $s_j = 0$, satisfies the following relation

$$\text{rank}\left[\frac{\partial S}{\partial x}\right] = K, \text{ for almost every } x$$

It will be established in Chapter 4 that the case $K < n$ corresponds to nonobservable systems. In this case, there exist analytic functions $g_1(x), \ldots,$ $g_{n-K}(x)$ such that the matrix

$$J = \frac{\partial(S, g_1, \ldots, g_{n-K})}{\partial x}$$

has full rank n. Then the system of equations

$$\left\{\begin{aligned} \tilde{x}_1 &= h_1(x, u, \ldots, u^{(\alpha)}) \\ &\vdots \\ \tilde{x}_{s_1} &= h_1^{(s_1-1)}(x, u, \ldots, u^{(\alpha+s_1-1)}) \\ \tilde{x}_{s_1+1} &= h_2(x, u, \ldots, u^{(\alpha)}) \\ &\vdots \\ \tilde{x}_{s_1+s_2} &= h_2^{(s_2-1)}(x, u, \ldots, u^{(\alpha+s_2-1)}) \\ &\vdots \\ \tilde{x}_{s_1+s_2+\ldots+s_p} &= h_p^{(s_p-1)}(x, u, \ldots, u^{(\alpha+s_p-1)}) \\ \tilde{x}_{s_1+s_2+\ldots+s_p+i} &= g_i(x, u, \ldots, u^{(\gamma)}) \qquad i = 1, \ldots, n-K \end{aligned}\right. \tag{2.3}$$

is of the form $F_i(x, \tilde{x}, u, \ldots, u^{(\gamma)}) = 0, i = 1, \ldots, n$ with

$$\partial(F_1, \ldots, F_n)/\partial(x_1, \ldots, x_n) = J$$

To avoid the introduction of new notations, it is not restrictive to assume $\gamma \geq \max\{\alpha+s_i-1, i = 1, \ldots, p\}$. The determinant of J is an analytic function whose set of zeros has an empty interior, so there exists an open dense subset V of $\mathbb{R}^{n+m\gamma}$ such that $\det J$ is different from zero at every point of V and the implicit function theorem applies. Therefore there exist n functions

$$x_i = \phi_i(\tilde{x}, u, \ldots, u^{(\gamma)}) \quad \text{for} \quad 1 \leq i \leq n$$

which define a local diffeomorphism ϕ parametrized by $u, \ldots, u^{(\gamma)}$:

$$x = \phi(\tilde{x}) \tag{2.4}$$

By applying the change of coordinates induced by (2.4), the system (2.1) becomes

$$\left\{\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= \tilde{x}_4 \\
&\vdots \\
\dot{\tilde{x}}_{s_1} &= h_1^{(s_1)}(\phi(\tilde{x}), u, \ldots, u^{(\gamma)}) \\
\dot{\tilde{x}}_{s_1+1} &= \tilde{x}_{s_1+2} \\
&\vdots \\
\dot{\tilde{x}}_{s_1+s_2} &= h_2^{(s_2)}(\phi(\tilde{x}), u, \ldots, u^{(\gamma)}) \\
&\vdots \\
\dot{\tilde{x}}_{s_1+\cdots+s_p} &= h_p^{(s_p)}(\phi(\tilde{x}), u, \ldots, u^{(\gamma)}) \\
\dot{\tilde{x}}_{s_1+\cdots+s_p+i} &= g_i(\tilde{x}), u, \ldots, u^{(\gamma)}) \qquad i = 1, \ldots, n-K \\
y_1 &= \tilde{x}_1 \\
y_2 &= \tilde{x}_{s_1+1} \\
&\vdots \\
y_p &= \tilde{x}_{s_1+\cdots+s_{p-1}+1}
\end{aligned}\right. \tag{2.5}$$

In the neighborhood of any point where $\det J \neq 0$, also

$$\frac{\partial h_i^{(s_i)}}{\partial x} \in \operatorname{span}_{\mathcal{K}} \left\{ \frac{\partial h_1}{\partial x}, \ldots, \frac{\partial h_1^{(s_1-1)}}{\partial x}, \frac{\partial h_2}{\partial x}, \ldots, \frac{\partial h_i^{(s_i-1)}}{\partial x} \right\}$$

so that

$$\begin{aligned}
\frac{\partial h_i^{(s_i)}}{\partial \tilde{x}} &= [c_1 \ \ldots \ c_{s_1+\cdots+s_i} \ 0 \ \ldots \ 0] J \frac{\partial x}{\partial \tilde{x}_j} \\
&= [c_1 \ \ldots \ c_{s_1+\cdots+s_i} \ 0 \ \ldots \ 0] \ e_j \ = \ 0 \qquad j \ > \ s_1 + \cdots + s_i
\end{aligned}$$

where e_j is the jth column of the identity matrix. Therefore the functions $h_i^{(s_i)}(\phi(\tilde{x}), u, \ldots, u^{(\gamma)})$ depend only on $\tilde{x}_1, \ldots, \tilde{x}_{s_1+\ldots+s_i}$.
Since the following identities hold,

$$\begin{aligned}
&y_1 = \tilde{x}_1, \\
&\dot{y}_1 = \tilde{x}_2, \ldots, \\
&y_1^{(r)} = \tilde{x}_{1+r} \quad \text{for } r = 0, \ldots, s_1 - 1 \\
&\vdots \\
&y_j = \tilde{x}_{s_1+\cdots+s_{j-1}+1} \\
&\dot{y}_j = \tilde{x}_{s_1+\cdots+s_{j-1}+2}, \ldots, \\
&y_j^{(r)} = \tilde{x}_{s_1+\cdots+s_{j-1}+1+r} \ \text{for } r = 0, \ldots, s_j - 1, \ j = 2, \ldots, p
\end{aligned}$$

From (2.5), we get the input-output relations we were looking for:

$$
\begin{aligned}
y_1^{(s_1)} &= h_1^{(s_1)}(\phi(y_1, \dot{y}_1, \ldots, y_1^{(s_1-1)}), u, \ldots, u^{(\gamma)}) \\
&\vdots \\
y_j^{(s_j)} &= h_j^{(s_j)}(\phi(y_1, \ldots, y_1^{(s_1-1)}, y_j, \ldots, y_j^{(s_j-1)}), u, \ldots, u^{(\gamma)}) \qquad (2.6) \\
&\vdots \\
y_p^{(s_p)} &= h_p^{(s_p)}(\phi(y_1, \ldots, y_1^{(s_1-1)}, \ldots, y_p, \ldots, y_p^{(s_p-1)}), u, \ldots, u^{(\gamma)})
\end{aligned}
$$

∎

The input-output equations (2.6) are not uniquely defined since, for instance, if K is less than n, different choices of the functions $g_i(x, u, \ldots, u^{(\gamma)})$ produce a different system (2.3).
Instead of $\{s_1, \ldots, s_p\}$, it is possible to use the observability indices as defined in Chapter 4 to derive an analogous input-output equation.

2.2 Examples

Example 2.2. For the system

$$
\begin{cases}
\dot{x}_1 = x_3 u_1 \\
\dot{x}_2 = u_1 \\
\dot{x}_3 = u_2 \\
y_1 = x_1 \\
y_2 = x_2
\end{cases}
$$

we have $\dot{y}_1 = x_3 u_1$, $\ddot{y}_1 = u_2 u_1 + x_3 \dot{u}_1$, and finally

$$\ddot{y}_1 = u_2 u_1 + (\dot{y}_1 / u_1)\dot{u}_1$$

The last equation holds at every point in which $u_1 \neq 0$. For the second output, $\dot{y}_2 = u_1$ immediately.

The following example shows that for a more general nonlinear system, where $\dot{x}$ does not appear explicitly, such as

$$F(x, \dot{x}, u, \ldots, u^{(\nu)}) \qquad (2.7)$$

the method described above cannot be applied.

Example 2.3. Consider the system

$$
\begin{cases}
(\dot{x} - u)^2 = 0 \\
y = x
\end{cases}
$$

The implicit function theorem cannot be invoked to obtain x, since for every x and every u, $\partial(\dot{x} - u)^2/\partial x = 0$. By the way, an input-output relation for this example is given by

$$(\dot{y} - u)^2 = 0$$

or by

$$(\dot{y} - u) = 0$$

Results similar to those described above may be found in [156]. A state elimination method which yields global results is studied in [44].

2.3 Generalized Realization

Let us consider now the problem of going from an external, input-output representation of a dynamic system to an internal, state-space representation, that, in a sense to be made precise, defines the same relation between inputs and outputs as the external representation. We are interested in what is called the realization problem. In general, as an outcome of the realization process, starting from an external representation, one would like to obtain internal, state-space representations of the form (1.4) or, at least, of the form (1.5). These will be termed *classical realizations*; more general representations, as those we will discuss in this section, will be termed *generalized realization.*
To begin with, we first recall some results from [54], which follow quite naturally from elementary manipulation of input-output equations and which yield a generalized realization. In the next section, a necessary and sufficient condition is given for the existence of a classical realization in the single-input/single-output (SISO) case.
Consider an input-output differential equation of the form

$$F(y,\ldots,y^{(k)},u,\ldots,u^{(s)})=0 \tag{2.8}$$

where u and y are, respectively, a scalar input and a scalar output, F is a meromorphic function of its arguments; and $\frac{\partial F}{\partial y^{(k)}}$ is generically nonzero. An internal representation of the system described by (2.8) can easily be constructed by introducing the new variable $x=(x_1,\ldots,x_k)$, defined by

$$(x_1,\ldots,x_k)=(y,\ldots,y^{(k-1)}) \tag{2.9}$$

This yields the following set of implicit state equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{k-1} = x_k \\ F(x_1,\ldots,x_k,\dot{x}_k,u,\ldots,u^{(s)}) = 0 \end{cases} \tag{2.10}$$

The assumption about $\frac{\partial F}{\partial y^{(k)}}$ and the implicit function theorem , now, allow us to write, at least locally,

$$\begin{cases} \dot{x}_1 = x_2 \\ \quad\vdots \\ \dot{x}_{k-1} = x_k \\ \dot{x}_k = \varphi(x,u,\dot{u},\ldots,u^{(s)}) \\ y = x_1 \end{cases} \tag{2.11}$$

Equations (2.11) give a representation of the input-output relation described by (2.8) with internal variables. Compared with (1.4), the representation given by (2.11) can be said to be a generalized realization; the adjective “generalized” accounts for the presence of derivatives of u. According to this, the variable x can be interpreted as a generalized state variable. In addition, note that the application of the implicit function theorem, beside being nonconstructive, does not guarantee that φ is a meromorphic function.

Example 2.4. Consider the input-output equation $\dot{y}^2 = y + u$. The above procedure yields the implicit state equations

$$\dot{x}^2 = x + u$$

or, locally, one of the following explicit realizations, depending on whether $\dot{y} > 0$ or $\dot{y} < 0$.

$$\begin{cases} \dot{x} = \sqrt{x+u} \\ y = x \end{cases} \tag{2.12}$$

$$\begin{cases} \dot{x} = -\sqrt{x+u} \\ y = x \end{cases} \tag{2.13}$$

Note that the right-hand side of the state equations in the above representations is not meromorphic at the origin.

In general, the above procedure does not yield classical realizations of the form (1.4). Also linear input-output relations, in case transmission zeros are present, give rise, in this way, to generalized realizations. It can be said that, in general, the presence of derivatives of u is somehow related to the presence of zero dynamics (this concept will be made more precise in Section 5.6, see also [88]) However, as we will show in the next section, generalized realizations of the form (2.11) may be transformed under suitable hypotheses into a realization containing no derivatives of u.

Example 2.5. Consider the linear input-output relation $\ddot{y} = u + \dot{u}$ that corresponds to the transfer function $\frac{s+1}{s^2}$, having a zero in $s = -1$. Although the input-output relation is linear, the above procedure yields a generalized realization:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -u - \dot{u} \\ y = x_1 \end{cases}$$

The notions of controllability/accessibility and of observability that one can use in characterizing the structure of internal representations are reported in Chapters 3 and 4. Without going into the details now, we mention that, with respect to those notions, realization (2.11) is in general observable, but not necessarily accessible . In this sense, it is not minimal.

2.4 Classical Realization

Conditions for ensuring the existence of a classical realization of the form (1.4), in particular containing no derivatives of the input, are fully characterized in [33] (see also [28, 47, 56, 91, 92, 150, 151, 158]). There exist simple input-output relations which do not admit a classical realization. A typical example in this sense is given by the input-output relation $\ddot{y} = \dot{u}^2$.
Here we introduce an elementary result that fully solves the problem for the input-output relations having the particular form

$$y^{(k)} = \varphi(y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)}). \tag{2.14}$$

where φ is a meromorphic function of its arguments and $\frac{\partial \varphi}{\partial y^{(k)}}$ is generically nonzero. The input-output relation (2.14) admits a realization if and only if the right-hand side of (4.17) has a special polynomial structure in the derivatives of u. To investigate this structure, consider the dynamic system Σ_e whose input is $u^{(s+1)}$ and whose state is $(y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)})$.

$$\Sigma_e : \frac{d}{dt} \begin{bmatrix} y \\ \vdots \\ y^{(k-2)} \\ y^{(k-1)} \\ u \\ \vdots \\ u^{(s-1)} \\ u^{(s)} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \vdots \\ y^{(k-1)} \\ \varphi \\ \dot{u} \\ \vdots \\ u^{(s)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u^{(s+1)} \tag{2.15}$$
$$= f_e + g_e u^{(s+1)}$$

Given system (2.14), define the field $\mathcal{K}$ of meromorphic functions in a finite number of variables y, u, and their time derivatives. Let $\mathcal{E}$ be the formal vector space $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$.

Define the following subspace of $\mathcal{E}$

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\}$$

Obviously, any one-form in $\mathcal{H}_1$ has to be differentiated at least once to depend explicitly on $\mathrm{d}u^{(s+1)}$. Let $\mathcal{H}_2$ denote the subspace of $\mathcal{E}$ which consists of all one-forms that have to be differentiated at least twice to depend explicitly on $\mathrm{d}u^{(s+1)}$. From (2.15), one easily computes

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-1)}\}$$

$\mathcal{H}_2$ is a subspace of $\mathcal{H}_1$ which is more generally computed as

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_1 \mid \dot{\omega} \in \mathcal{H}_1\} \tag{2.16}$$

More generally, define $\mathcal{H}_i$ as the subspace of $\mathcal{E}$ which consists of all one-forms that have to be differentiated at least i times to depend explicitly on $\mathrm{d}u^{(s+1)}$.

More precisely, the subspaces $\mathcal{H}_i$ are defined by induction as follows for $i \geq 2$.

$$\mathcal{H}_{i+1} = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_i \mid \dot{\omega} \in \mathcal{H}_i\}$$

These subspaces will be used extensively later in this book and especially in Chapter 3.

2.5 Input-output Equivalence and Realizations

To introduce the equivalence of input-output systems and to study their minimal state-space realization, we will use systems Σ_e, as defined in (2.15). Consider $\mathcal{H}_\infty^* = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_1^* \mid \omega^{(k)} \in \mathcal{H}_1^*, \forall k \geq 0\} = 0$. Each nonzero vector in $\mathcal{H}_\infty^*$ is said to be *autonomous* for system (2.14).

2.5.1 Irreducible Input-output Systems

In this section, we will formalize a *reduction* algorithm to obtain the notion of input-output equivalence and a definition of realization.

Definition 2.6 (Irreducible input-output system). *System (2.14) is said to be an* irreducible input-output system *if the associated system (2.15) satisfies*

$$\mathcal{H}_\infty = 0$$

Example 2.7. The input-output system $\ddot{y} = yu^2 + y\dot{u}$ is irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ \dot{y} \\ u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ yu^2 + y\dot{u} \\ \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ddot{u}$$

is such that $\mathcal{H}_\infty = 0$. It is worth noting that the set of solutions $(u(t), y(t))$ of $\dot{y} = yu$ is a subset of the set of solutions of $\ddot{y} = yu^2 + y\dot{u}$, but the systems are not "equivalent" according to the forthcoming Definition 2.13.

Example 2.8. $\ddot{y} = \dot{u} + (\dot{y} - u)^2$ is not irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ \dot{y} \\ u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \dot{u} + (\dot{y} - u)^2 \\ \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ddot{u}$$

is not irreducible since $\mathrm{d}(\dot{y} - u) \in \mathcal{H}_\infty$ and we will claim that $\dot{y} = u$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y} - u)^2$.

2.5.2 Reduced Differential Form

We are interested in minimal realizations, *i.e.* of the lowest order. We introduce definitions of *reduced differential form, reduced input-output system and irreducible differential form, etc.* to reach that goal.

Definition 2.9 (Reduced differential form). *An exact form* $\mathrm{d}\phi' \in \mathcal{H}_\infty$ *is said to be a* reduced differential form *of system (2.14) if*

(a) $\mathrm{d}\phi' \not\equiv 0$
(b) $\mathrm{d}\phi' \in \mathcal{H}_\infty$.

Definition 2.10 (Reduced input-output system). *Let* $\mathrm{d}\phi'$ *be a reduced differential form, that produces the differential equation*

$$\phi'(y, \cdots, y^{(k'-1)}, y^{(k')}, u, \cdots, u^{(s')}) = 0 \tag{2.17}$$

such that $\partial\phi'/\partial y^{(k')} \neq 0$, $\partial\phi'/\partial u^{(s')} \neq 0$, *and* $\partial^2\phi'/\partial {y^{(k')}}^2 \equiv 0$ *with* $k' > 0$, $s' \geq 0$. *Equation (2.17) has a unique solution under the condition* $\partial\phi'/\partial y^{(k')} \not\equiv 0$

$$y^{(k')} = \varphi'(y, \cdots, y^{(k'-1)}, u, \cdots, u^{(s')}) \tag{2.18}$$

Then (2.18) is called a reduced input-output system *of system (2.14).*

Definition 2.11 (Irreducible differential form). *If (2.18) is an irreducible input-output system in the sense of Definition 2.6, then* $\mathrm{d}(y^{(k')} - \varphi')$ *is said to be an irreducible differential form of (2.14).*

Example 2.12 (Example 2.8 cont'd). $\mathrm{d}(\dot{y}-u) \in \mathcal{H}_\infty$ and $\dot{y} = u$ is an irreducible system. Thus, $\phi' = \dot{y} - u = 0$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y} - u)^2$.

It is not true that any input-output system has an irreducible input-output system. Consider

$$\ddot{y} = \frac{\dot{y}\dot{u}}{u} \tag{2.19}$$

$d\phi' = \mathrm{d}(\dot{y}/u)$ is a reduced differential form of (2.19) according to Definition 2.9. Thus, system (2.19) is not irreducible. Let $\phi' = \dot{y}/u = 0$, which is not an *irreducible input-output system* in the sense of the above Definition. Therefore, system (2.19) does not admit any irreducible input-output system.

In the special case of linear time-invariant systems, the reduction procedure corresponds to a pole/zero cancellation in the transfer function. For nonlinear systems, the above procedure also generalizes the so-called *primitive step* in [28].

2.5.3 Input-output Equivalence

We restrict our attention to the family of input-output systems that admit an irreducible input-output system: see Definition 2.6. Therefore, it is possible to introduce an equivalence relation on the family.

Definition 2.13 (Input-output equivalence). *Two input-output systems are said to be input-output equivalent if they have the same irreducible input-output system representation*

$$y^{(\kappa)} = \varphi(y, \dots, y^{(\kappa-1)}, u, \dots, u^{(\sigma)}) \tag{2.20}$$

Example 2.14. The two systems

$$\ddot{y} = \dot{u} - 2(\dot{y} - u)^2$$

and

$$y^{(3)} = \ddot{u}$$

do admit the same irreducible input-output system, $\dot{y} = u$.

2.5.4 Realizations

A general definition of *realization* is given, that describes the relationships between state-space equations (1.1) and input-output equations (2.14).

An algorithm realizing the state-space systems (1.1) from input-output systems (2.14) will be provided in Section 2.8.1, as well as a necessary and sufficient condition for the existence of such a realization.

Definition 2.15 (Realization). *A state-space system (1.1) is said to be a realization of the input-output system (2.14) if the elimination of the state variables in (1.1) yields an input-output equation described by*

$$y^{(\kappa)} = \phi(y, \dots, y^{(\kappa-1)}, u, \dots, u^{(\sigma)})$$

which is input-output equivalent to (2.14).

The system (2.14) is said to be realizable if there exists a realization in the sense of Definition 2.15.

2.6 A Necessary and Sufficient Condition for the Existence of a Realization

We make use here of the subspaces introduced in (2.16), to derive a full characterization of the existence of a classical realization.

Theorem 2.16. *There exists an observable state-space system*

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{2.21}$$

which is a realization for (2.14) if and only if

- $k > s$
- *and,* $\mathcal{H}_i$ *is integrable for each* $i = 1, \dots, s+2$.

Proof. Sufficiency: Let $\{\mathrm{d}\xi_1, \dots, \mathrm{d}\xi_k\}$ be a basis of $\mathcal{H}_{s+2}$. From the construction of the subspaces $\mathcal{H}_i$,

$$\begin{aligned} \mathcal{H}_{s+1} &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\} \\ \mathcal{H}_s &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}\dot{u}\} \\ &\vdots \\ \mathcal{H}_1 &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \dots, \mathrm{d}u^{(s)}\} \end{aligned} \tag{2.22}$$

Introduce the following coordinate transformation for the system (2.15):

$$\begin{aligned} x_1 &= \xi_1(y, \dot{y}, \dots, u^{(s)}) \\ &\vdots \\ x_k &= \xi_k(y, \dot{y}, \dots, u^{(s)}) \\ x_{k+1} &= u \\ &\vdots \\ x_{k+s+1} &= u^{(s)} \end{aligned} \tag{2.23}$$

From $\mathcal{H}_{s+2} \subset \mathcal{H}_{s+1}$, it follows $\mathrm{d}\dot{\xi}_i = \sum_{j=1}^{k} \alpha \mathrm{d}\xi + \beta \mathrm{d}u$, for each $j = 1, \dots, k$. Let $x = (x_1, \dots, x_k)$. Thus, at least locally,

$$\dot{x} = f(x,u) \tag{2.24}$$

The assumption $k > s$ indicates that the output y depends only on x.
Necessity: Assume that the observable state-space system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

is a realization for the input-output system (2.14). Since the state-space system is proper, necessarily $k > s$.

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{(s)}\} \\ &\vdots \\ \mathcal{H}_{s+1} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\} \\ \mathcal{H}_{s+2} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \end{aligned}$$

From (2.23), the spaces $\mathcal{H}_i$ are integrable as expected. ∎

Example 2.17. Let $\ddot{y} = \dot{u}^2$, and compute

$$\begin{aligned}\mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u, \mathrm{d}\dot{u}\}\\ \mathcal{H}_2 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u\}\\ \mathcal{H}_3 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u)\}\end{aligned}$$

Since $\mathcal{H}_3$ is not integrable, there does not exist any state-space system generating $\ddot{y} = \dot{u}^2$. This can be checked directly, or using some results in [33].

Example 2.18. Let $\ddot{y} = u^2$. The conditions of Theorem 2.16 are fulfilled and the state variables $x_1 = y$ and $x_2 = \dot{y}$ yield

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u^2\\ y &= x_1\end{aligned}$$

whose state elimination yields $\ddot{y} = u^2$.

2.7 Minimal Realizations

The notion of minimality here is standard for linear systems and means that the dimension of the state-space system equals the order of some reduced transfer function.

A minimal realization can be obtained directly from the input-output equation. The notion of irreducible form is used as it is for linear time-invariant systems. A minimal realization is obtained when constructing a realization as in the proof of Theorem 2.16, or applying the algorithm in Section 2.8.1, to an irreducible input-output system, whenever it exists. More precisely, one has

Theorem 2.19. *Given an input-output system (2.14), assume that the conditions in Theorem 2.16 are fulfilled. Then, there exists an observable and controllable, i.e., minimal, realization of order k for (2.14), if and only if (2.14) is an irreducible input-output system.*

Proof. Given (2.14), the generating system (2.21) obtained from Theorem 2.16 is observable. The extended system (2.15) can be written in the coordinates (2.23). It then reads as the composite system of system (2.24) and the controllable string of integrators $\dot{u}^{(i)} = u^{(i+1)}$, $i = 0, ..., s$. Thus, (2.15) is accessible if and only if (2.24) is controllable. The result of Theorem 2.19 follows since (2.15) is controllable if and only if (2.14) is irreducible, by Definition 2.6. ∎

Example 2.20. Consider

$$\phi = \ddot{y} - \dot{y}u - y\dot{u}.$$

Compute $f_e = \begin{pmatrix} \dot{y} \\ \dot{y}u + y\dot{u} \\ \dot{u} \\ 0 \end{pmatrix}$ and $g_e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Thus, $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu) \in \mathcal{H}_\infty$. An irreducible differential form of $\phi = 0$ is $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu)$. A minimal realization is obtained for $\phi = 0$ as

$$\begin{cases} \dot{x} = xu \\ y = x \end{cases}$$

2.8 Affine Realizations

2.8.1 A Realization Algorithm

Under the conditions of Theorem 2.16, any basis of $\mathcal{H}_{s+2}$ defines a state space of the input-output system (2.14). The purpose of this section is to give an algorithmic construction of a canonical affine state-space representation; it results from a special choice of the basis for $\mathcal{H}_{s+2}$ under some special structure of the input-output equation. Consider the input-output equation (2.14).

Algorithm 2.21

Step 1.
Let $r := k - s$, *then* $\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}\}$ *is a basis for*

$$\mathcal{X}_1 := \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{(j)}, j \geq 0\}$$

- *If* $\partial^2\varphi/\partial(u^{(s)})^2 \neq 0$, *stop!*
- *If* $\partial^2\varphi/\partial(u^{(s)})^2 = 0$ *and* $\mathrm{d}(\partial y^{(k)}/\partial u^{(s)}) \neq 0$, *define*

$$y_{11} = \partial y^{(k)}/\partial u^{(s)} \tag{2.25}$$

If $\mathrm{d}(y^{(r)} - \frac{\partial y^{(k)}}{\partial u^{(s)}}u) \neq 0$, *define*

$$y_{12} = y^{(r)} - \frac{\partial y^{(k)}}{\partial u^{(s)}}u \tag{2.26}$$

y_{11} *and* y_{12} *are called* auxiliary *outputs.*

Step 2.
- *If* $\mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{11}^{(i)}, i \geq 0\} = 0$, *then stop!*
- *Let* $\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}; \mathrm{d}y_{11}, \ldots, \mathrm{d}y_{11}^{(r_{11}-1)}\}$ *be a basis for*

$$\mathcal{X}_{21} := \mathcal{X}_1 + \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{11}^{(i)}, i \geq 0\}$$

where $r_{11} = \dim\mathcal{X}_{21} - \dim\mathcal{X}_1$.
- *If* $\mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{12}^{(i)}, i \geq 0\} = 0$, *then stop!*
- *Let* $\{\mathrm{d}y, \ldots, \mathrm{d}y_{11}^{(r_{11}-1)}; \mathrm{d}y_{12}, \ldots, \mathrm{d}y_{12}^{(r_{12}-1)}\}$ *be a basis for*

$$\mathcal{X}_2 := \mathcal{X}_{21} + \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{12}^{(i)}, i \geq 0\}$$

where $r_{12} = \dim\mathcal{X}_2 - \dim\mathcal{X}_{21}$.

• *If* $\forall \ell \geq r_{1j}$, $\mathrm{d}y_{1j}^{(\ell)} \in \mathcal{X}_2$, *set* $s_{1j} = -1$, *for* $j = 1, 2$.

If $\exists \ell \geq r_{1j}, \mathrm{d}y_{1j}^{(\ell)} \notin \mathcal{X}_2$, *then define* $s_{1j} \geq 0$ *as the smallest integer such that, abusing the notation, one has locally*

$$y_{1j}^{(r_{1j}+s_{1j})} = y_{1j}^{(r_{1j}+s_{1j})}(y^{(\lambda)}, y_{11}^{(\sigma_{11})}, y_{12}^{(\sigma_{12})}, u, \ldots, u^{(s_{1j})})$$

where $0 \leq \lambda < r$, $0 \leq \sigma_{11} < r_{11} + s_{11}$, $0 \leq \sigma_{12} < r_{12} + s_{12}$.

• *If* $s_{11} \geq 0$ *and* $\partial^2 y_{11}^{(r_{11}+s_{11})} / \partial (u^{(s_{11})})^2 \neq 0$

or if $s_{12} \geq 0$ *and* $\partial^2 y_{12}^{(r_{12}+s_{12})} / \partial (u^{(s_{12})})^2 \neq 0$ *stop!*

• *If* $\mathcal{X}_2 + \mathcal{U} = \mathcal{Y} + \mathcal{U}$, *and* $\partial^2 y_{1j}^{(r_{1j}+s_{1j})} / \partial (u^{(s_{1j})})^2 = 0$ *whenever* $s_{1j} \geq 0$, *then the algorithm stops and the realization is complete. Otherwise, define the new* auxiliary *outputs, whenever* $\mathrm{d}(\partial y_{1j}^{(r_{1j}+s_{1j})} / \partial u^{(s_{1j})}) \neq 0$, *respectively,* $\mathrm{d}(y_{1j}^{(r_{1j})} - \dfrac{\partial y_{1j}^{(r_{1j}+s_{1j})}}{\partial u^{(s_{1j})}} u) \neq 0$*:*

$$\begin{aligned} y_{21} &= \frac{\partial y_{11}^{(r_{11}+s_{11})}}{\partial u^{(s_{11})}} \\ y_{22} &= y_{11}^{(r_{11})} - \frac{\partial y_{11}^{(r_{11}+s_{11})}}{\partial u^{(s_{11})}} u \\ y_{23} &= \frac{\partial y_{12}^{(r_{12}+s_{12})}}{\partial u^{(s_{12})}} \\ y_{24} &= y_{12}^{(r_{12})} - \frac{\partial y_{12}^{(r_{12}+s_{12})}}{\partial u^{(s_{12})}} u \end{aligned}$$

Step i+1.

From Step i, one has defined a set of numbers $r_{i-1,j}$ *and* $s_{i-1,j}$ *as well as the auxiliary outputs*

$$\begin{aligned} y_{i,2j-1} &= \partial y_{i-1,j}^{(r_{i-1,j}+s_{i-1,j})} / \partial u^{(s_{i-1,j})} \\ y_{i,2j} &= y_{i-1,j}^{(r_{i-1,j})} - \frac{\partial y_{i-1,j}^{(r_{i-1,j}+s_{i-1,j})}}{\partial u^{(s_{i-1,j})}} u \end{aligned} \tag{2.27}$$

for some $j \in \{1, \cdots, 2^{i-1}\}$.

• *If* $\mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{i1}^{(\ell)}, \ell \geq 0\} = 0$, *then stop!*

• *Let* $\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}; \ldots; \mathrm{d}y_{i1}, \ldots, \mathrm{d}y_{i1}^{(r_{i1}-1)}\}$ *be a basis for*

$$\mathcal{X}_{i+1,1} := \mathcal{X}_i + \mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{i1}^{(\ell)}, \ell \geq 0\}$$

where $r_{i1} = \dim\mathcal{X}_{i+1,1} - \dim\mathcal{X}_i$.

• *If* $\mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{ij}^{(\ell)}, \ell \geq 0\} = 0$ *for* $j = 2, \ldots, 2^{i-1}$, *then stop!*

• *For* $j = 2, \ldots, 2^{i-1}$, *let*

$$\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}; \ldots; \mathrm{d}y_{ij}, \ldots, \mathrm{d}y_{ij}^{(r_{ij}-1)}\}$$

be a basis for

$$\mathcal{X}_{i+1,j} := \mathcal{X}_{i+1,j-1} + \mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{ij}^{(\ell)}, \ell \geq 0\}$$

where $r_{ij} = \mathrm{dim}\mathcal{X}_{i+1,j} - \mathrm{dim}\mathcal{X}_{i+1,j-1}$. *Set* $\mathcal{X}_{i+1} = \sum \mathcal{X}_{i+1,j}$

• *If* $\forall \ell \geq r_{ij}, \mathrm{d}y_{ij}^{(r_{ij})} \in \mathcal{X}_{i+1}$, *set* $s_{ij} = -1$.

If $\exists \ell \geq r_{ij}, \mathrm{d}y_{ij}^{(\ell)} \notin \mathcal{X}_{i+1}$, *then define* s_{ij} *as the smallest integer such that, abusing the notation, one has locally*

$$y_{ij}^{(r_{ij}+s_{ij})} = y_{ij}^{(r_{ij}+s_{ij})}(y^{(\lambda)}, y_{ij}^{(\sigma)}, u, \ldots, u^{(s_{ij})})$$

where $0 < \lambda < r$, $0 < \sigma < r_{ij} + s_{ij}$.

• *If* $s_{ij} \geq 0$ *and* $\partial^2 y_{ij}^{(r_{ij}+s_{ij})} / \partial u^{(s_{ij})^2} \neq 0$ *for some* $j = 1, \ldots, 2^{i-1}$, *stop!*

• *If* $\mathcal{X}_{i+1} + \mathcal{U} = \mathcal{Y} + \mathcal{U}$ *and* $\partial^2 y_{ij}^{(r_{ij}+s_{ij})} / \partial u^{(s_{ij})^2} = 0$, *whenever* $s_{ij} \geq 0$, *then the algorithm stops and the realization is completed. Otherwise, define the new* auxiliary *outputs, whenever* $\mathrm{d}(\partial y_{ij}^{(r_{ij}+s_{ij})} / \partial u^{(s_{ij})}) \neq 0$, *respectively,* $\mathrm{d}(y_{ij}^{(r_{ij})} - \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}} u) \neq 0$*:*

$$y_{i+1,2j-1} = \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}}, \; y_{i+1,2j} = y_{ij}^{(r_{ij})} - \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}} u$$

End of the algorithm.

Algorithm 2.21 yields the definition of the state $(x_1, \ldots, x_k) = (y^{(\lambda)}, y_{ij}^{(\sigma)})$ where $0 < \lambda < r$, $0 < \sigma < r_{ij} + s_{ij}$. General necessary and sufficient conditions for the existence of an affine state representation are derived from the algorithm as well.

Theorem 2.22. *System (2.14) admits an affine realization if and only if Algorithm 2.21 can be completed, or equivalently,*

- $k > s$ *and*

$$\frac{\partial^2 y^{(k)}}{\partial (u^{(s)})^2} = 0 \tag{2.28}$$

- *for* $s_{ij} \geq 0$ *and any* $r_{ij} > 0$, $i = 1, 2, \ldots, N$, $j = 1, \ldots, 2^i$,

$$\frac{\partial^2 y_{ij}^{(r_{ij}+s_{ij})}}{\partial (u^{(s_{ij})})^2} = 0 \tag{2.29}$$

  where y_{ij}, r_{ij}, *and* s_{ij} *are as defined in Algorithm 2.21,*
- *there exists a finite integer* $N \geq 1$ *such that*

$$\sum_{i=1}^{N} \mathcal{X}_i + \mathcal{U} = \mathcal{Y} + \mathcal{U} \tag{2.30}$$

Remark 2.23. Condition (2.29) is mentioned in [28, 158]. It embodies the fact that the input-output equation (2.14) as well as the differential equations relating the auxiliary outputs are affine in the highest time derivative of the input.

Proof (Proof of Theorem 2.22).
Sufficiency: Algorithm 2.21 can be performed if conditions (2.28) and (2.29) are satisfied. State variables are defined in the procedure of the algorithm. This algorithm will be completed in finite steps according to condition (2.30). Consequently, an affine, observable generating system is obtained for the input-output system (2.14).
Necessity: To prove the necessity condition we need a lemma, which is partly contained in [28, 29, 158].

Lemma 2.24. *If there exists a state-space system*

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{2.31}$$

which is a generating system for (2.14), locally around any point $(y_0, ..., u_0^{(s)})$ *in some suitable open dense subset of* $\mathbb{R}^{k+s+1}$*, then* $\partial^2 y^{(k)}/\partial {u^{(s)}}^2 = 0$*,* $\mathrm{d}y_{11} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$*, and* $\mathrm{d}y_{12} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$*.*

Proof. It is already known that $\partial^2 y^{(k)}/\partial {u^{(s)}}^2 = 0$ is a necessary condition for the existence of an affine realization of a given input-output system [28, 29, 158].

The rest of the statement follows from the equality

$$\begin{aligned} y^{(k-s)} &= L_f^{k-s} h + [L_g L_f^{k-s-1} h]u \\ &= y_{12} + y_{11}u \end{aligned}$$

∎

If there exists an affine realization, then it can be transformed into the canonical structure displayed by Algorithm 2.21. By Lemma 2.24, (2.28) holds. Condition (2.29) follows from the proof of Lemma 2.24 which is applied to each auxiliary output y_{ij}, considering all state variables in $\mathcal{X}_{i-1}$ as parameters. The realization is observable and the dimension of the state-space is finite, which imply (2.30). ∎

2.8.2 Examples

Example 2.25.

Given the input-output differential equation

$$\ddot{y} = u^2 \sin y \cos y + \dot{u} \sin y \tag{2.32}$$

for which $k = 2$ and $s = 1$, define

$$x_1 = y^{(k-s-1)} = y$$

Let

$$y_{11} = \sin y, \qquad y_{12} = \dot{y} - u \sin y$$

Then $k_{21} = 0$, $k_{22} = 1$. The relation

$$\dot{y}_{12} = -y_{12} u \cos y$$

implies that $s_{22} = 0$. Define

$$x_2 = y_{12}^{(k_{22}-s_{22}-1)} = y_{12}$$

Then a realization of (2.32) is obtained:

$$\begin{cases} \dot{x}_1 = x_2 + (\sin x_1)\, u \\ \dot{x}_2 = -x_2(\cos x_1)\, u \\ y = x_1 \end{cases} \tag{2.33}$$

which is both observable and accessible and therefore it is minimal.

Example 2.26.

Consider the input-output system:

$$u\ddot{y} - u\dot{y}(u^2 - \dot{y}^2)^{1/2} - \dot{y}\dot{u} = 0 \tag{2.34}$$

and write it as

$$\ddot{y} = \dot{y}(u^2 - \dot{y}^2)^{1/2} + \frac{\dot{y}}{u}\dot{u} \tag{2.35}$$

The right-hand side of (2.35) is meromorphic on the open and dense subset of $I\!R^3$, containing the points $(\dot{y}, u, \dot{u})$ such that $u^2 > \dot{y}^2$. Use Algorithm 2.21 to define

$$x_1 = y^{(k-s-1)} = y$$

and define the auxiliary outputs:

$$y_{11} = \frac{\dot{y}}{u}, \qquad y_{12} = \dot{y} - \frac{\dot{y}}{u} u = 0$$

Then,

$$\dot{y}_{11} = y_{11}(1 - y_{11}^2)^{1/2} u$$

Define

$$x_2 = y_{11}$$

A realization is obtained which has the representation:

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = x_2(1 - x_2^2)^{1/2} u \\ y = x_1 \end{cases} \tag{2.36}$$

It does not satisfy the strong accessibility rank condition, so it is not a minimal realization.

Example 2.27. [27]

The input-output system

$$\begin{aligned} y^2y^{(3)}u^2 - y^3u^4 - (3\dot{y}/y + 2\dot{u}/u)\ddot{y}y^2u^2 + 2\dot{y}^3u^2 \\ +2\dot{y}\dot{u}^2y^2 + 2\dot{y}^2\dot{u}yu - \dot{y}\ddot{u}y^2u = 0 \end{aligned} \tag{2.37}$$

can be written as

$$\begin{aligned} y^{(3)} = {} & yu^2 + (3\dot{y}/y + 2\dot{u}/u)\ddot{y} - 2\dot{y}^3/y^2 - 2\dot{y}\dot{u}^2/u^2 \\ & -2\dot{y}^2\dot{u}/(yu) + \dot{y}\ddot{u}/u \end{aligned} \tag{2.38}$$

and has been considered before (see Example 2 of [27]). From Step 1 of Algorithm 2.21, $k = 3$, $s = 2$. Let $x_1 = y$ and define $y_{11} = \dot{y}/u$. Then in Step 2 of the algorithm,

$$\ddot{y}_{11} = yu + 3y_{11}\dot{y}_{11}u/y - 2y_{11}^3u^2/y^2 + y_{11}^2\dot{u}/y.$$

So, $k_{11} = 2$ and $s_{11} = 1$. Let $x_2 = y_{11}$ and define

$$y_{22} = \dot{y}_{11} - y_{11}^2u/y$$

Then

$$\dot{y}_{22} = (y + y_{11}y_{22}/y)u$$

Thus $x_1 = y$, $x_2 = y_{11}$, and $x_3 = y_{22}$ yield

$$\begin{cases} \dot{x}_1 = x_2u \\ \dot{x}_2 = x_3 + (x_2^2/x_1)u \\ \dot{x}_3 = (x_1 + x_2x_3/x_1)u \\ \;\, y = x_1 \end{cases} \tag{2.39}$$

Realization (2.39) is different from the realization given in [27] which is not required to fit within the canonical scheme of Algorithm 2.21.

2.9 The Hopping Robot

Consider a hopping robot consisting of a body and a single leg, as sketched in Figure 2.1 . The orientation of the body with respect to the leg is actuated through torque u_1. The length of the leg may vary with the translation of a piston and it is controlled through a force u_2. Although the realization theory was developed for single input systems, it can easily be used to consider this two-input system. It is modeled as follows. Let m be the mass of the leg, J the inertia momentum of the body, r the (variable) length of the leg, θ denotes the angular position of the body, and ϕ the angular position of the leg.

If the action of gravity is neglected, then the mechanical equations yield

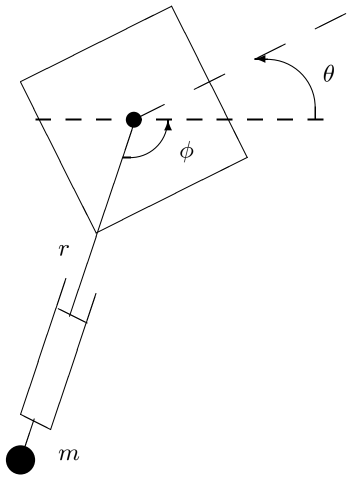

Fig. 2.1. Hopping robot

$$\begin{aligned} m\ddot{r} - mr\dot{\phi}^2 &= u_2 \\ J\ddot{\theta} &= u_1 \\ mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} &= -u_1 \end{aligned} \tag{2.40}$$

Equations (2.40) are higher order input/ouput equations, considering the three outputs (r, θ, ϕ). Construct the extended system Σ_e defined in (2.15).

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\phi}^2 \\ \dot{\theta} \\ 0 \\ \dot{\phi} \\ -2\frac{\dot{r}\dot{\phi}}{r} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/m \\ 0 & 0 \\ 1/J & 0 \\ 0 & 0 \\ -\frac{1}{mr^2} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2.41}$$

The latter is not accessible and $\mathcal{H}_\infty$ is spanned by $(2mr\dot{\phi}\mathrm{d}r + mr^2\mathrm{d}\dot{\phi} + J\mathrm{d}\dot{\theta})$. this one-form is exact and equals $\mathrm{d}(mr^2\dot{\phi} + J\dot{\theta})$. This is the kinetic momentum of the hopping robot and is constant. Its minimal realization has not dimension 6. A reduced input-output representation is obtained by

$$\begin{aligned} m\ddot{r} - mr\dot{\phi}^2 &= u_2 \\ mr^2\dot{\phi} + J\dot{\theta} &= 0 \\ mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} &= -u_1 \end{aligned} \tag{2.42}$$

Apply the procedure again, compute the new extended system Σ_e, whose dimension is 5 now, and check $\mathcal{H}_\infty = /$. A minimal realization of the hopping robot (without gravity) thus has dimension 5. Suitable state variables may be chosen as $(r, \dot{r}, \theta, \phi, \dot{\phi})$.

2.10 Some Models

2.10.1 Electromechanical Systems

Consider an inverted pendulum of length l with a point mass m attached at the end of the beam, which is actuated by the torque u applied at the base of the beam. Let g denote the gravitational constant, and φ the angular position of the pendulum with respect to the vertical position. Then the equation of motion is

$$ml^2\ddot{\varphi} - mgl\sin\varphi = u.$$

The angle φ is the output. Rewriting it into the form (2.14),

$$\ddot{\varphi} = \frac{1}{ml^2}[u + mgl\sin\varphi].$$

Algorithm 2.21 can be applied that yields the obvious state variables $x_1 = \varphi$ and $x_2 = \dot{\varphi}$. The state realization is then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{ml^2}[u + mgl\sin x_1] \\ \varphi &= x_1 \end{aligned}$$

2.10.2 Virus Dynamics

Several models of virus dynamics can be found in [129]. Let us consider here the HIV infection and the elementary modeling of the immune system when it is subject to HIV infection. The immune system is based on two main actors, the so-called CD8 cells and the CD4 cells. The CD4 cells act as markers, they mark and identify the undesirable agents as viruses, bacteria, etc. The CD8 cells act as killers. However the CD8 cells kill only agents that have been marked beforehand by some CD4 cell. The body is subject to many infectious agents, and the majority of those infections have no consequence at all. Some of them are agressive against specific tissues of the body and the immune system is able to eliminate the infection. What is unfortunate about HIV is that this virus attacks the basis of the immune system itself. The HIV virus infects CD4 cells which will no longer be able to mark the HIV virions. After the population of healthy CD4 cells decreases, the HIV virions will thus be protected against the immune system. Infected CD4 cells act as host cells and they produce new HIV virions. An elementary model may be derived. Let T denote the population of healthy CD4 cells. Let T^* denote the population of infected CD4 cells. Let v denote the population of HIV virions. As any living specie, the CD4 cells have some finite lifetime $1/\delta$. The evolution of some *independent* population is then approximated by the linear first-order system:

$$\dot{T} = -\delta T$$

The body is assumed to produce new CD4 cells at some constant rate s; thus, the evolution of T in a noninfected body will be described by

$$\dot{T} = s - \delta T$$

and the population T stabilizes at some equilibrium $T_0 = s/\delta$. In an infected body, besides natural death, the population T decreases due to the agression of the virus. Part of the healthy CD4 cells will be converted into infected CD4 cells. It is supposed to be proportional to both the T population and the v population. Finally, the dynamics of T is

$$\dot{T} = s - \delta T - \beta T v$$

The population T^* of infected CD4 cells is also submitted to a natural death, with a lifetime $1/\mu$. The only source of production of new infected CD4 cells has already been described and its rate equals $\beta T v$. Thus, the dynamics of T^* reads as

$$\dot{T}^* = \beta T v - \mu T^*$$

The population v of HIV virions is submitted to a natural death and their lifetime equals $1/c$. The production of new virions is proportional to the population T^* of infected CD4 cells. Let us exclude here the case of new external injection of some virus load. Then the dynamics of v becomes

$$\dot{v} = kT^* - cv$$

Problems

2.1. Consider the following "Ball and Beam" system [166], whose input is the angle α and whose output is the ball position r. The input-output equation of the system is

$$0 = \left(\frac{J}{R^2} + m\right)\ddot{r} + mg\sin\alpha - mr\dot{\alpha}^2$$

where the constant parameters J, R, m, g represent, respectively, the inertia of the ball, its radius, its mass, and the gravitational constant.

1. Write a generalized state space representation of the system, if any.
2. Write a classical state-space realization, if any. Hint: Apply Theorem 2.16.

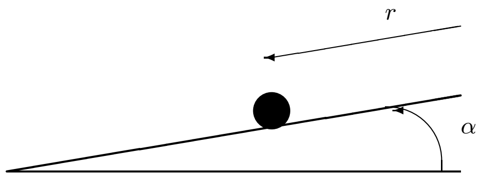

Fig. 2.2. Ball and Beam

2.2. Consider the same "ball and beam" system as in Exercise 2.1 and assume that the angle α is produced by a torque u, so that

$$\ddot{\alpha} = u$$

Considering u as the input and r as the output, write a classical state-space realization.

2.3. Consider the following "Pendulum on a cart" system. Let m and l be the

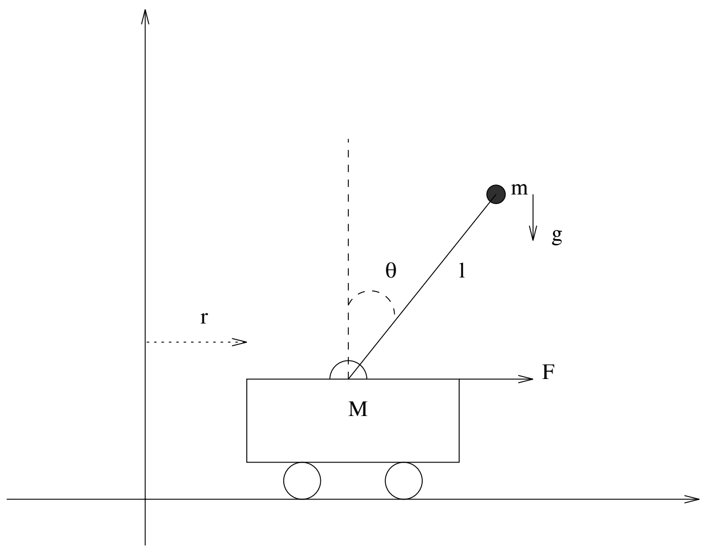

Fig. 2.3. Pendulum on a cart

mass and the length of the pendulum, let M be the mass of the cart. The external force F applied to the cart is the control variable. This system can be modeled as

$$\begin{aligned}(M+m)\ddot{r} + b\dot{r} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= F\\ (I+ml^2)\ddot{\theta} + mgl\sin\theta &= -ml\ddot{r}\cos\theta\end{aligned}$$

Considering the output $y = \theta$, write a classical state-space realization, if any.

3

Accessibility

3.1 Introduction

A basic notion in control systems theory is that of reachable state and controllability. Controllability concerns the possibility of steering the system from a state x_1 to another state x_2. For linear systems, controllability is a structural property, in the sense that any linear system can be split into a controllable subsystem and an autonomous one.

This is not the case for nonlinear systems, as the examples in Section 3.2 show. The structural property that in the nonlinear case plays a role similar to that of controllability in the linear case and can be given a similar characterization is the accessibility property.

To illustrate the notions of reachability, controllability, and accessibility, we will consider, in this chapter, a system without outputs of the form

$$\dot{x} = f(x) + g(x)u \tag{3.1}$$

where $x \in I\!R^n$, $u \in I\!R^m$, and the entries of $f(x)$ and $g(x)$ are elements of $\mathcal{K}$.

3.2 Examples

The following examples illustrate some classical pathologies stemming from the fact that, in general, one cannot expect a nonlinear system to be controllable from any initial state to any final state.

Example 3.1. Consider the following one-dimensional system

$$\dot{x} = xu \tag{3.2}$$

This system is clearly not controllable from zero: if the initial state is located at the origin, then the trajectory $x(0, u(t))$ of (3.2) remains at the origin for any input function $u(t)$.

From an initial state x_0 different from the origin, it is not possible to reach a state x_1 such that $x_1 x_0 < 0$ since, for any continuous function u(t), the trajectory should pass through the origin and remain there. However, any point x_1 such that $x_1 x_0 > 0$ is reachable from $x_0 \neq 0$.

Example 3.2.

$$\begin{aligned} \dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= u \end{aligned} \tag{3.3}$$

This system is not controllable from any initial point x_0, in the sense that any neighborhood of x_0 contains a point x_1 which cannot be reached from x_0. Figure 3.1 below shows the set of points which are reachable from x_0.

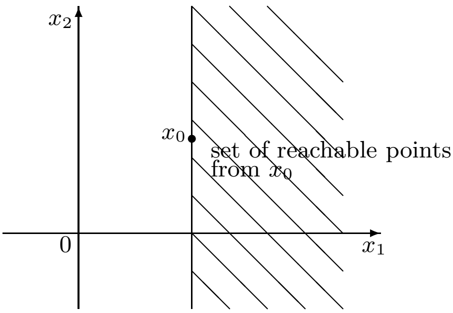

Fig. 3.1. Reachable set of system (3.3)

3.3 Reachability, Controllability, and Accessibility

The following definitions formalize the phenomena displayed by the introductory examples.

Definition 3.3. *For a nonlinear system Σ of the form (3.1), the state x_1 is said to be reachable from the state x_0 if there exists a finite time T and a Lebesgue measurable function $u(t) : [0, T] \to \mathbb{R}^m$, such that $x(x_0, u, T) = x_1$.*

For the system (3.2), the state $x_1 = 2$ is reachable from $x_0 = 1$, whereas $x_2 = -1$ is not.

Definition 3.4. *A system Σ of the form (3.1) is said to be controllable at x_0 if there exists a neighborhood V of x_0, such that any state x_1 in V is reachable from x_0.*

The system (3.2) is controllable at any state different from the origin, whereas there does not exist any state x_0 in $I\!R^2$ at which the system (3.3) is controllable.

Definition 3.5. *A system Σ of the form (3.1) is said to be accessible at x_0 if the set of reachable points from x_0 contains an open subset of $I\!R^n$.*

As illustrated in Figure 3.1, the system (3.3) is accessible in this sense at any point of $I\!R^2$.

From [154], the accessibility of a nonlinear analytic system at a given point x_0 of the state space $I\!R^n$ is characterized by the fact that the so-called strong accessibility distribution $\mathcal{L}$ has dimension n at that point. The existence of an open and dense submanifold of the state space $I\!R^n$ at whose points the system is accessible is then characterized by the fact that the strong accessibility distribution $\mathcal{L}$ has generically dimension n .

From an algebraic point of view, accessibility will be characterized in the next Section using the concepts of *autonomous element* introduced in [136].

3.4 Autonomous Elements

Let $\mathcal{X}$ denote the subspace of $\mathcal{E}$ given by $\mathcal{X} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_i, i = 1, \dots, n\}$.

Definition 3.6. *A nonzero function φ in $\mathcal{K}_\Sigma$ is said to be an autonomous element of a system Σ of the form (3.1) if there exists an integer ν and a nonzero meromorphic function F so that*

$$F(\varphi, \dot{\varphi}, ..., \varphi^{(\nu)}) = 0 \tag{3.4}$$

where $\dot{\varphi} = \delta\varphi$.

Definition 3.7. *Let φ be a function in $\mathcal{K}_\Sigma$ such that $\mathrm{d}\varphi \in \mathcal{X}$. The relative degree r of φ is given by*

$$r = inf\{k \in I\!N,\ such\ that\ \mathrm{d}\varphi^{(k)} \notin \mathcal{X}\}. \tag{3.5}$$

In particular, we say that φ has finite relative degree if r belongs to $I\!N$ and that φ has infinite relative degree if $r = \infty$.

Remark 3.8. Note that if φ has relative degree equal to k, then

$$\operatorname{span}_{\mathcal{K}}\{\mathrm{d}\varphi, ..., \mathrm{d}\varphi^{(k-1)}\} \subset \mathcal{X}$$

$$\operatorname{span}_{\mathcal{K}}\{\mathrm{d}\varphi, ..., \mathrm{d}\varphi^{(k)}\} \not\subset \mathcal{X}$$

Proposition 3.9. *If a function φ in $\mathcal{K}_\Sigma$ is an autonomous element for a system Σ of the form (3.1), then*

(i) $d\varphi \in \mathcal{X}$

(ii)φ has infinite relative degree.

Proof. Any vector $\omega \in \mathcal{E}$ and which does not belong to $\mathcal{X}$ satisfies

$$\dim \text{span}_{\mathcal{K}}\{\omega, ..., \omega^{(k-1)}\} = k \tag{3.6}$$

for any $k \geq 1$. From Definition 3.6, this is not true for $\omega = \mathrm{d}\varphi$ and $k = \nu + 1$. This ends the proof of statement (i).
If φ has a finite relative degree, then

$$\dim \text{span}_{\mathcal{K}}\{\mathrm{d}\varphi, ..., \mathrm{d}\varphi^{(k-1)}\} = k \tag{3.7}$$

for any $k \geq 1$. This contradicts Definition 3.6. ∎

The notion of autonomous element can be defined also in the context of non-exact forms.

Definition 3.10. *A one-form ω in $\mathcal{X}$ is said to be an autonomous element of a system Σ of the form (3.1) if there exists an integer ν and meromorphic function coefficients α_i in $\mathcal{K}$, for $i = 1, \ldots, \nu$, so that*

$$\alpha_0\omega + \ldots + \alpha_\nu\omega^{(\nu)} = 0 \tag{3.8}$$

Definition 3.11. *The relative degree r of a one-form ω in $\mathcal{X}$ is given by*

$$r = min\{k \in I\!N, |\ \text{span}_{\mathcal{K}}\{\omega, ..., \omega^{(k)}\} \not\subset \mathcal{X}\} \tag{3.9}$$

Proposition 3.12. *A one form ω in $\mathcal{X}$ is an autonomous element if and only if it has an infinite relative degree.*

Proof. Necessity: Assume that ω in $\mathcal{X}$ has an infinite relative degree. Since $\dim \mathcal{X} = n$, there exists $k, 0 \leq k \leq n$ such that

$$\dim \text{span}_{\mathcal{K}}\{\omega, ..., \omega^{(k)}\} = k \tag{3.10}$$

This yields that ω is autonomous.
Sufficiency: By contradiction, show that if ω has finite relative degree, then it is not autonomous. As a matter of fact, if ω has finite relative degree, then

$$\dim \text{span}_{\mathcal{K}}\{\omega, ..., \omega^{(k-1)}\} = k \tag{3.11}$$

for any $k \geq 1$. This completes the proof. ∎

It is now straightforward to prove

Proposition 3.13. *The function $\varphi \in \mathcal{K}$ and the one-form* $\mathrm{d}\varphi$ *have the same relative degree.*

As a consequence, φ is autonomous if and only if $d\varphi$ is autonomous.

Proposition 3.14. *The set $\mathcal{A}$ of autonomous elements of $\mathcal{E}$ is a subspace of $\mathcal{E}$.*

Proof. Using Proposition 3.12, the proof becomes straightforward. Consider two vectors in $\mathcal{A}$; their sum still has an infinite relative degree. The same holds for the product of an element in $\mathcal{A}$ and a scalar function in $\mathcal{K}_\Sigma$. ∎

3.5 Accessible Systems

Now, let us state formally the following definition:

Definition 3.15. *The system (3.1) is said to satisfy the strong accessibility condition if*

$$\mathcal{A} = 0 \tag{3.12}$$

or, equivalently, there does not exist any nonzero autonomous element in $\mathcal{K}$.

A practical criterion for evaluating accessibility is given as follows.

Accessibility Criterion: Computation of $\mathcal{A}$

Let us define a filtration of $\mathcal{E}$, *i.e.*, a sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ such that each $\mathcal{H}_k$, for $k > 0$, is the set of all one-forms with relative degree at least k.

The sequence is defined by induction as follows:

$$\begin{aligned}\mathcal{H}_0 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\},\\ \mathcal{H}_j &= \{\omega \in \mathcal{H}_{j-1} \mid \ \dot{\omega} \in \mathcal{H}_{j-1}\}.\end{aligned}$$

It is clear that this sequence is decreasing, *i.e.*, $\mathcal{E} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots$, and that, at the first step,

$$\mathcal{H}_1 \quad = \quad \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$$

An easy consequence of the construction is the following.

Proposition 3.16. *$\mathcal{H}_k$ is the space of one-forms whose relative degrees are greater than or equal to k. Furthermore, there exists an integer $k^* > 0$ such that:*

$$\mathcal{H}_k \supset \mathcal{H}_{k+1} \quad \text{for } k \leq k^*,$$

$$\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots = \mathcal{H}_\infty$$

$$\mathcal{H}_{k^*} \supsetneq \ \mathcal{H}_\infty$$

By definition, it follows that $\mathcal{A} = \mathcal{H}_\infty$. The existence of the integer k^* comes from the fact that each $\mathcal{H}_k$ is a finite-dimensional $\mathcal{K}$-vector space so that, at each step either the dimension decreases by at least one or $\mathcal{H}_{k+1} = \mathcal{H}_k$.

Systems that satisfy the strong accessibility condition get an easy algebraic characterization now [3].

Theorem 3.17. *The system (3.1) satisfies the strong accessibility condition if and only if*

$$\mathcal{H}_\infty = 0 \tag{3.13}$$

The condition (3.13) is locally equivalent to the fact that the strong accessibility distribution $\mathcal{L}$ spans the whole tangent bundle TM to the state manifold M, where the strong accessibility distribution $\mathcal{L}$ is defined as the limit of a filtration

$$0 \subset \Delta_1 \subset ... \subset \Delta_k \subset ... \subset TM$$

of *involutive* distributions Δ_k given by

$$\Delta_k = \overline{g + ad_f g + ... + ad_f^k g}$$

The remarkable fact is that the condition given by Theorem 3.17 does not require us to work with *exact* forms only. The practical construction of $\mathcal{H}_k$ is easier than that of Δ_k, since a low number of purely algebraic computations is required and no involutivity condition need to be considered.

3.6 Controllability Canonical Form

Although $\mathcal{H}_k$ is in general not closed, *i.e.*, it does not admit a basis which consists only of closed forms, the limit $\mathcal{A} = \mathcal{H}_\infty$ turns out to be closed. This follows from the fact that locally

$$\mathcal{A} \perp \mathcal{L}$$

and it is stated in the following Proposition ([86]):

Proposition 3.18. *Let* $\{\omega_1, ..., \omega_r\}$ *be a basis for* A, *then*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_m = 0, \;\; 1 \leq i \leq r \tag{3.14}$$

From the Frobenius Theorem, there exist locally r functions, say $\xi_1, ..., \xi_r$, with infinite relative degree so that

$$\mathcal{A} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1, ..., \mathrm{d}\xi_r\}$$

Since $\mathcal{A}$ is invariant under time differentiation, in particular,

$$\begin{aligned} \dot{\xi}_1 &= f_1(\xi_1, \cdots, \xi_r) \\ &\vdots \\ \dot{\xi}_r &= f_r(\xi_1, \cdots, \xi_r) \end{aligned} \tag{3.15}$$

Now, choosing $n - r$ arbitrary functions, say $\xi_{r+1}, \cdots, \xi_n$, so that

$$\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1, ..., \mathrm{d}\xi_n\}$$

where $\mathcal{X}$ denotes $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$, one derives a representation, called the controllability canonical form, of system (3.1) of the form

$$\begin{aligned}
\dot{\xi}_1 &= f_1(\xi_1, \cdots, \xi_r) \\
&\vdots \\
\dot{\xi}_r &= f_r(\xi_1, \cdots, \xi_r) \\
\dot{\xi}_{r+1} &= f_{r+1}(\xi_1, \cdots, \xi_n) + g_{r+1}(\xi_1, \cdots, \xi_n)u \\
&\vdots \\
\dot{\xi}_n &= f_n(\xi_1, \cdots, \xi_n) + g_n(\xi_1, \cdots, \xi_n)u
\end{aligned} \tag{3.16}$$

Note that, as a consequence of the state elimination results in Section 2.1, the $\xi_1, \ldots, \xi_r$ are autonomous elements that satisfy a differential equation of order less than or equal to r.

3.7 Controllability Indices

Define, now,

$$h_i = \dim \mathcal{H}_i - \dim \mathcal{H}_{i+1}, \text{ for } i \geq 1$$

Moreover, h_{k^*} is nonzero and $h_k = 0$, for any $k > k^*$.

The set of controllability indices $\{k_1^*, \ldots, k_m^*\}$ of system (3.1) is defined as the dual set of $\{h_0, \ldots, h_{k^*}\}$ by the relations

$$h_i = \text{card}\{k_j^* | k_j^* \geq i\}$$

$$k_j^* = \text{card}\{h_i | h_i \geq j\} \text{ for } j = 1, \ldots, m$$

In particular, $k^* = \max\{k_1^*, \ldots, k_m^*\}$

Proposition 3.19. *For system (3.1),*

$$k_1^* + \ldots + k_m^* = n - \dim \mathcal{A} \tag{3.17}$$

Note that in the nonlinear setting, the controllability indices describe only the *structure* of the accessible subsystem. It is not possible in general to display them as in the linear case by a Brunovsky canonical form.

Example 3.20. Consider the unicycle described in Figure 3.2 below whose state representation is

$$\dot{x} = \begin{bmatrix} \cos x_3 \; u_1 \\ \sin x_3 \; u_1 \\ u_2 \end{bmatrix}.$$

Compute

$$\begin{aligned}
\mathcal{H}_1 &= \text{span}_{\mathcal{K}}\{\mathrm{d}x\} \\
\mathcal{H}_2 &= \text{span}_{\mathcal{K}}\{(\sin x_3)\mathrm{d}x_1 - (\cos x_3)\mathrm{d}x_2\} \\
\mathcal{H}_3 &= 0
\end{aligned}$$

The controllability indices are computed as follows. $h_1 = 2$, $h_2 = 1$, $h_3 = 0, \ldots$ and $k_1^* = 2$, $k_2^* = 1$. However, there does not exist any change of coordinates that gives rise to a representation containing a Brunovsky block of dimension 2. The system is accessible; there does not exist any autonomous element.

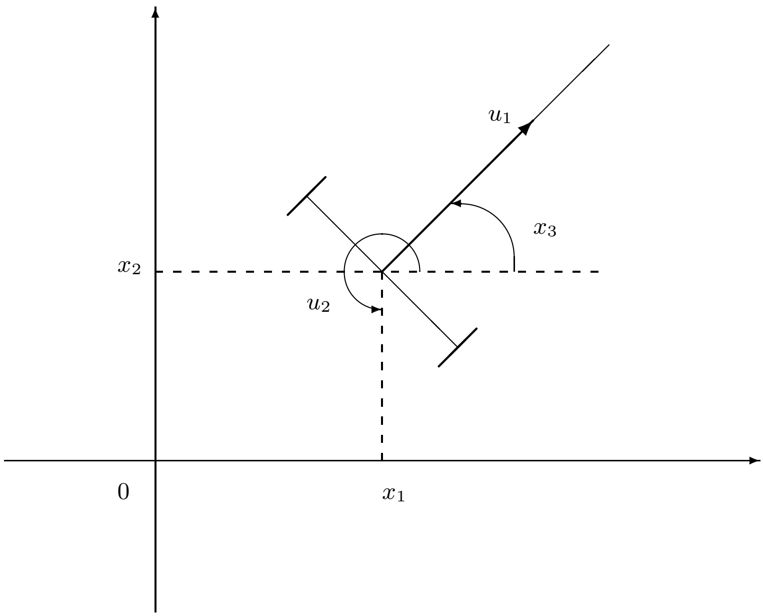

Fig. 3.2. The unicycle

Example 3.21. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_3 & 0 \\ x_4 & 0 \\ \vdots & \vdots \\ x_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.18}$$

Then, compute

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{x_3 \mathrm{d}x_1 - \mathrm{d}x_2, \ldots, x_n \mathrm{d}x_1 - \mathrm{d}x_{n-1}\}$$

and more generally, for $2 \leq k \leq n-1$,

$$\mathcal{H}_k = \mathrm{span}_{\mathcal{K}}\{x_3 \mathrm{d}x_1 - \mathrm{d}x_2, \ldots, x_{n-k+2} \mathrm{d}x_1 - \mathrm{d}x_{n-k+1}\}$$

$$\mathcal{H}_{n-1} = \mathrm{span}_{\mathcal{K}}\{x_3 \mathrm{d}x_1 - \mathrm{d}x_2\}$$

$$\mathcal{H}_n = \mathcal{H}_\infty = 0$$

Thus, $h_1 = 2$, $h_2 = 1$, $h_3 = 1, \ldots$, $h_{n_1} = 1$, $h_n = 0$ and $k_1^* = n-1$, $k_2^* = 1$

3.8 Accessibility of the Hopping Robot Model

Consider again the hopping robot of Section 2.9.

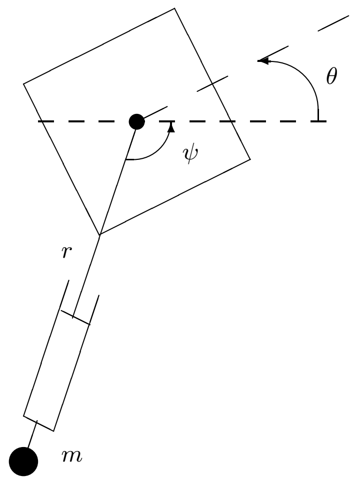

Fig. 3.3. Hopping robot

Choosing the state variables as $(x_1, ..., x_6) = (l, \dot{l}, \theta, \dot{\theta}, \psi, \dot{\psi})$, the resulting state equations are

$$\dot{x} = \begin{pmatrix} x_2 \\ x_1 x_6^2 \\ x_4 \\ 0 \\ x_6 \\ -2\frac{x_2 x_6}{x_1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/m \\ 0 & 0 \\ 1/J & 0 \\ 0 & 0 \\ -\frac{1}{m x_1^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.19}$$

The system (3.19) is not accessible, because, as remarked previously, $\mathcal{H}_\infty$ is spanned by $(2mx_1x_6\mathrm{d}x_1 + mx_1^2\mathrm{d}x_6 + J\mathrm{d}x_4)$. The kinetic momentum $mx_1^2x_6 + Jx_4$ of the hopping robot is constant and it represents a *noncontrollable* component of the state.

Problems

3.1. Consider the realization of the ball and beam system obtained from Exercise 2.2 and check its accessibility.

3.2. Compute the controllability indices of the hopping robot (3.19).

3.3. Consider the linear system $\dot{x} = Ax + Bu$. Compute the spaces $\mathcal{H}_k$, for $k \geq 1$, in terms of the matrices A and B and derive the standard controllability criterion for linear systems.

4

Observability

4.1 Introduction

The notion of the observability of a linear or nonlinear dynamic system concerns the possibility of recovering the state $x(t)$ from knowledge of the measured output $y(t)$, the input $u(t)$, and, possibly, a finite number of their time derivatives $y^{(k)}(t),\ k \geq 0$, and $u^{(l)}(t),\ l \geq 0$. The structural property which can be easily characterized in a nonlinear framework concerns the existence of an open and dense submanifold of the state space $I\!R^n$ around whose points the system is locally observable. Thus, the situation is quite similar to the one pertaining to controllability.

To illustrate the notions of observability we will consider systems of the form (1.4), that is,

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $y \in I\!R^p$, and the entries of f, g and h are meromorphic functions of x. We will sometimes ignore the presence of inputs in the system.

4.2 Examples

Example 4.1. Let us consider the system Σ defined by

$$\begin{cases} \dot{x} = 0 \\ y = x^2 \end{cases} \tag{4.1}$$

Clearly, for this system, it is not possible to distinguish between positive and negative values of the state just from knowledge of the output. We will say, in a situation like this, that Σ is not observable in a neighborhood of the origin. If one has the additional information that the state x belongs to an open neighborhood of some point x_0 which does not contain the origin, then its value can be deduced from the value of the output y by

$$x = \mathrm{sgn}(x_0)\sqrt{y}$$

where sgn(.) denotes the sign function. We will say, then, that Σ (4.1) is locally observable around any point different from the origin.

Example 4.2. Let q be an integer, $q \geq 1$, and let us consider the system Σ defined by

$$\begin{cases} \dot{x} = x^{2q} \\ y = x^2 \end{cases} \tag{4.2}$$

As in Example (4.1), the knowledge of y at time t is not sufficient to deduce the value of x at time t. However, $\dot{y} = 2x^{2q+1}$ and so,

$$x = \begin{cases} 0, & \text{if } y = 0 \\ \frac{\dot{y}}{2y^q}, & \text{if } y \neq 0 \end{cases}$$

In this case, we will say that Σ is observable.

Further discussions and examples about the notion of observability for nonlinear systems may be found in [45, 73, 131, 163, 164].

4.3 Observability

In terms of the possibility of recovering the state, the notion that gets a nice structural characterization in the nonlinear context is the so-called *local weak observability*. Such a notion is defined using the concept of (in-)distinguishable states [73].

Definition 4.3. *Given a system Σ of the form (1.4), two states x^1 and x^2 in X are said to be indistinguishable if, for any admissible input function $u(t)$ and any initial time t_0, the outputs $y^1(t)$ and $y^2(t)$ corresponding to $u(t)$ and to the initial condition $x(t_0) = x^1$ or, respectively, $x(t_0) = x^2$ are equal for any $t \geq t_0$.*

Example 4.4. Consider the unicycle in Example 3.20:

$$\dot{x} = \begin{bmatrix} \cos x_3 \; u_1 \\ \sin x_3 \; u_1 \\ u_2 \end{bmatrix}$$

with output $y = x_1$. Let $x^1 = (0, 0, 0)$ and $x^2 = (0, z, 0)$. The states x^1 and x^2 are indistinguishable for any $z \in \mathbb{R}$ since

$$y^1(t_0, x^1, u(t)) = y^2(t_0, x^2, u(t)) = \int_{t_0}^{t} \cos\left(\int_{t_0}^{\tau} u_2(\sigma)\mathrm{d}\sigma\right) u_1(\tau)\mathrm{d}\tau$$

The above definition is scarcely useful for practical purposes, since it is based on a condition difficult to check. It can, however, be strengthened and made easier to handle by imposing that the considered input functions keep the state trajectory inside a given subset of the state space. One gets, in this way, the following definition:

Definition 4.5. *Given a system Σ of the form (1.4), let V be a subset of the state space X. Two states $x^1 \in V$ and $x^2 \in V$ are said to be V-indistinguishable if, for any admissible input function $u(t)$, for which the corresponding state trajectories originating from x^1 or from x^2 remain in V, the corresponding outputs $y^1(t)$ and $y^2(t)$ are equal for any $t \geq t_0$*

The above definition of V-indistinguishability does not induce an equivalence relation on the state space X (as the notion of indistinguishability actually does) since it is obviously not transitive. Let us denote, however, by $I_V(x^0)$ the set of states which are indistinguishable from x_0. Then, we can state the definition.

Definition 4.6. *Given a system Σ of the form (1.4), a state $x^0 \in X$ is locally weakly observable if there exists an open region $M \subseteq X$ containing x^0, such that, for any open neighborhood V of x^0 contained in M, $I_V(x^0) = \{x^0\}$*

Definition 4.7. *A system Σ of the form (1.4) is said to be locally weakly observable if there exists an open dense subset M of its state space X, such that any $x^0 \in M$ is locally weakly observable.*

In the rest of this book, local weak observability will simply be termed observability. From a practical point of view, this property expresses the fact that the state x can be recovered as a function of the output y, the input u, and a finite number of their time derivatives.

4.4 The Observable Space

Given a system Σ of the form (1.4), let us denote by $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ the spaces defined, respectively, by $\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$, $\mathcal{U} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u^{(j)}, j \geq 0\}$, and $\mathcal{Y} = \cup_{i\geq 0}\mathcal{Y}^i$, where $\mathcal{Y}^i = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{(j)}, 0 \leq j \leq i\}$. The chain of subspaces

$$0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \ldots \subset \mathcal{O}_k \subset \ldots \tag{4.3}$$

where $\mathcal{O}_k := \mathcal{X} \cap (\mathcal{Y}^k + \mathcal{U})$ is called the observability filtration.
In the special case of linear systems, $\mathcal{O}_k$ reads as

$$\mathcal{O}_k = \mathrm{span}_{\mathcal{K}}\{C\mathrm{d}x, CA\mathrm{d}x, \ldots, CA^k\mathrm{d}x\}$$

If we denote by $\mathcal{O}_\infty$ the limit of the observability filtration, it is easy to see that

$$\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$$

and we can state the following definition.

Definition 4.8. *The subspace $\mathcal{O}_\infty \subseteq \mathcal{E}$ is called the observable space of the system Σ.*

The following theorem (whose proof is left for an exercise) describes the first basic property of the subspace $\mathcal{O}_\infty$

Theorem 4.9. *The observable subspace $\mathcal{O}_\infty$ of Σ is such that*

$$dim\ \mathcal{O}_\infty = \mathrm{rank}_{\mathcal{K}} \left[\frac{\partial(y, \dot{y}, \dots, y^{(n-1)})}{\partial x}\right] \tag{4.4}$$

4.4.1 An Observability Criterion

The general result characterizing the observability of a system Σ of the form (1.4) follows from [73].

Theorem 4.10. *A system Σ of the form (1.4) is observable if and only if*

$$\mathcal{O}_\infty = \mathcal{X} \tag{4.5}$$

As a consequence of Theorems 4.9 and 4.10, we obtain the following observability rank condition.

Corollary 4.11. *A system Σ of the form (1.4) is observable if and only if*

$$\mathrm{rank}_{\mathcal{K}} \left[\frac{\partial(y, \dot{y}, \dots, y^{(n-1)})}{\partial x}\right] = n$$

The above rank condition reduces to the standard Kalman criterion for observability in the special case of linear systems.

Example 4.12. Consider the system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 + x_3 u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \ \ y = x_1 \end{cases}$$

for which $\dot{y} = x_2 + x_3 u$ and $y^{(2)} = x_1 + x_3 \dot{u} + x_2 u$. Computation shows that

$$\begin{aligned} \mathcal{O}_0 &= \mathcal{X} \cap (\mathcal{Y}^0 + \mathcal{U}) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1\} \\ \mathcal{O}_1 &= \mathcal{X} \cap (\mathcal{Y}^1 + \mathcal{U}) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2 + u\mathrm{d}x_3\} \\ \mathcal{O}_2 &= \mathcal{X} \cap (\mathcal{Y}^2 + \mathcal{U}) = \mathcal{X} \end{aligned}$$

Then, the system is observable, since $\mathcal{O}_\infty = \mathcal{O}_2 = \mathcal{X}$. Alternatively,

$$\mathrm{rank}_{\mathcal{K}} \left[\frac{\partial(y, \dot{y}, y^{(2)})}{\partial x}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 1 & u & \dot{u} \end{bmatrix} = 3$$

which yields the same results.

A further basic property of $\mathcal{O}_\infty$, as a space of differential forms, is integrability. This is shown, without loss of generality, in the case $p = m = 1$ in the next theorem.

Theorem 4.13. *The observability space $\mathcal{O}_\infty$ of a system Σ of the form (1.4) is closed.*

Proof. Without loss of generality, assume that $p = m = 1$. Compute the successive time derivatives of the output y as follows.

$$y = h(x) = a_0^0(x) \tag{4.6}$$

$\dot{y}$ is a polynomial in u:

$$\dot{y} = a_0^1(x) + a_1^1(x)u \tag{4.7}$$

$\ddot{y}$ is a polynomial in $u, \dot{u}$:

$$\ddot{y} = \sum_{\substack{ij \leq 2 \\ 1 \leq i \leq 2 \\ j \leq 2}} a_{ij}^2(x)u^{(i-1)^j} \tag{4.8}$$

More generally, $y^{(s)}$ is a polynomial in $u^{(i_1-1)^{j_1}} \cdots u^{(i_k-1)^{j_k}}$, for $i_1j_1 + \cdots + i_kj_k \leq s$, $1 \leq i \leq s$, $j \leq s$:

$$y^{(s)} = \sum_{\substack{i_1j_1 + \cdots + i_kj_k \leq s \\ 1 \leq i \leq s \\ j \leq s}} a_{i_1j_1\ldots i_kj_k}^s(x)u^{(i_1-1_1^j)} \cdots u^{(i_k-1_k^j)} \tag{4.9}$$

and finally, compute $y^{(n-1)}$ as a polynomial in $u^{(i_1-1)^{j_1}} \cdots u^{(i_k-1)^{j_k}}$, for $i_1j_1 + \cdots + i_kj_k \leq n-1$, $1 \leq i \leq n-1$, $j \leq n-1$:

$$y^{(n-1)} = \sum_{\substack{i_1j_1 + \cdots + i_kj_k \leq s \\ 1 \leq i \leq n-1 \\ j \leq n-1}} a_{i_1j_1\ldots i_kj_k}^{n-1}(x)u^{(i_1-1_1^j)} \cdots u^{(i_k-1_k^j)} \tag{4.10}$$

There are at most n independent coefficients $a_{i\cdots j}^\ell$ since

$$\operatorname{rank}\frac{\partial(a_0^0(x), \ldots, a_{i\cdots j}^\ell(x), \cdots)}{\partial x} \leq n$$

Denote $c_1(x), \ldots, c_\nu(x)$ those independent coefficients. Then, for any index, $a_{i\cdots j}^\ell$ may be expressed as a function $\varphi_{i\cdots j}^\ell$ of $c_1(x), \ldots, c_\nu(x)$. Substitute those expressions in (4.6), (4.7), and (4.10). The resulting systems can then be solved in $c_1(x), \ldots, c_\nu(x)$:

$$c_i(x) = C_i(y, \dot{y}, \ldots, y^{(n-1)}, u, \ldots, u^{(n-2)})$$

Finally,

$$\mathcal{O}_\infty = \mathrm{span}\{\mathrm{d}c_1(x), \ldots, \mathrm{d}c_\nu(x)\}$$

∎

Example 4.14. Consider the system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 x_4 + x_3 u \\ \dot{x}_2 = x_2 \\ \dot{x}_3 = 0 \\ \dot{x}_4 = 0 \\ \ \ y = x_1 \end{cases}$$

for which $\dot{y} = x_2 x_4 + x_3 u$, $y^{(2)} = x_2 x_4 + x_3 \dot{u}$, $y^{(3)} = x_2 x_4 + x_3 u^{(2)}$ and, more generally, $y^{(k)} = x_2 x_4 + x_3 u^{(k)}$.
Computation shows that

$$\begin{aligned}
\mathcal{O}_0 &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1\} \\
\mathcal{O}_1 &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4) + u\mathrm{d}x_3\} \\
\mathcal{O}_2 &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4) + u\mathrm{d}x_3, \mathrm{d}(x_2x_4) + \dot{u}\mathrm{d}x_3\} \\
&= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4), \mathrm{d}x_3\} \\
\mathcal{O}_3 &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4) + u\mathrm{d}x_3, \mathrm{d}(x_2x_4) + \dot{u}\mathrm{d}x_3, \mathrm{d}(x_2x_4) + u^2\mathrm{d}x_3\} \\
&= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4), \mathrm{d}x_3\} \\
&\ \vdots \\
\mathcal{O}_k &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4) + u\mathrm{d}x_3, \mathrm{d}(x_2x_4) + \dot{u}\mathrm{d}x_3, \ldots, \mathrm{d}(x_2x_4) + u^k\mathrm{d}x_3\} \\
&= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_1, \mathrm{d}(x_2x_4), \mathrm{d}x_3\}
\end{aligned}$$

Then, the system is not observable, since $\mathcal{O}_\infty = \mathcal{O}_2 \neq \mathcal{X}$. However, letting $a_0^0(x) = x_1$, $a_0^1(x) = x_2x_4$, $a_1^1(x) = x_3$, $a_0^2(x) = x_2x_4$, $\ldots$

$$\mathrm{rank}\left[\frac{\partial(a_0^0(x), \ldots, a_{i\cdots j}^\ell(x), \cdots)}{\partial x}\right] = \mathrm{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x_4 & 0 & x_2 \\ 0 & 0 & 1 & 0 \\ 0 & x_4 & 0 & x_2 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = 3$$

Choosing, for instance, $c_1(x) = x_1 = y$, $c_2(x) = x_2x_4 = \dot{y} - \dfrac{\dot{y} - y^{(2)}}{u - \dot{u}}$, $c_3(x) = x_3 = \dfrac{\dot{y} - y^{(2)}}{u - \dot{u}}$, we can also write $\mathcal{O}_\infty = \mathrm{span}\{\mathrm{d}c_1(x), \mathrm{d}c_2(x), \mathrm{d}c_3(x)\}$.

4.5 Observability Canonical Form

Given the system Σ of the form (1.4), its observable space $\mathcal{O}_\infty$ has the invariance property described in the next proposition.

Proposition 4.15. *Given a system Σ of the form (1.4), $\dot{\mathcal{O}_\infty} \subseteq \mathcal{O}_\infty + \mathcal{U}$.*

Proof. Assume that $\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y}^k + \mathcal{U})$ and, hence, $\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y}^{k+1} + \mathcal{U})$ for all $k \geq 0$. Then, $w \in \mathcal{O}_\infty$ implies $w \in X$ and $w \in (\mathcal{Y}^k + \mathcal{U})$. So, $\dot{w} \in (\mathcal{X} + \mathcal{U})$ and $\dot{w} \in (\mathcal{Y}^{k+1} + \mathcal{U})$ or, in other terms $\dot{w} = w_\mathcal{X} + w_\mathcal{U} = w_\mathcal{Y} + w'_\mathcal{U}$, with $w_\mathcal{X} \in \mathcal{X}$, $w_\mathcal{U} \in \mathcal{U}$, $w'_\mathcal{U} \in \mathcal{U}$, $w_\mathcal{Y} \in \mathcal{Y}^{k+1}$. The above equality implies that $w_\mathcal{X}$ also belongs to $(\mathcal{Y}^{k+1} + \mathcal{U})$, then it belongs to $\mathcal{O}_\infty$, and, as a consequence, $\dot{w} = w_\mathcal{X} + w_\mathcal{U}$ belongs to $\mathcal{O}_\infty + \mathcal{U}$. ∎

Since $\mathcal{O}_\infty$ is closed, it has a basis of the form $\{\mathrm{d}z_1, \ldots, \mathrm{d}z_r\}$. Completing the set $\{z_1, \ldots, z_r\}$ to a basis $\{z_1, \ldots, z_r, z_{r+1}, \ldots, z_n\}$ of $I\!R^n$, it is easy to see, thanks to the invariance property shown above, that the system Σ, in these coordinates, reads as

$$\begin{cases} \dot{z}_1 = f_1(z_1, \ldots, z_r) + g_1(z_1, \ldots, z_r)u \\ \quad\vdots \\ \dot{z}_r = f_r(z_1, \ldots, z_r) + g_r(z_1, \ldots, z_r)u \\ \dot{z}_{r+1} = f_{r+1}(z) + g_{r+1}(z)u \\ \quad\vdots \\ \dot{z}_n = f_n(z) + g_n(z)u \\ y = h(z_1, \ldots, z_r) \end{cases} \tag{4.11}$$

We will call a representation of the form (4.11) a canonical form with respect to observability.

4.6 Observability Indices

Given a system Σ of the form (1.4), its observability filtration $0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \ldots \subset \mathcal{O}_k \subset \ldots$ defines a set of structural indices in the following way. Let the indices σ_i for $i \geq 1$ be defined by

$$\begin{aligned} \sigma_1 &= \dim \mathcal{O}_0 \\ \sigma_i &= \dim \tfrac{\mathcal{O}_{i-1}}{\mathcal{O}_{i-2}} \end{aligned}$$

for $i \geq 2$. The set of indices s_1, for $i \geq 1$, which is dual to the set $\{\sigma_i, i \geq 1\}$, is defined by

$$s_i = \mathrm{card}\{\sigma_j \text{ such that } \sigma_j \geq i\} \tag{4.12}$$

The integer σ_j represents the number of observability indices s_i which are greater than or equal to j, and duality implies that $\sigma_j = \mathrm{card}\{s_i \text{ such that } s_i \geq j\}$.

Definition 4.16. *Given a system Σ of the form (1.4), the set of indices $\{s_1, \ldots, s_p\}$ defined by (4.12) is called the set of observability indices of Σ.*

The key property of observability indices we are interested in is described in the following proposition.

Proposition 4.17. *Given a system Σ of the form (1.4), one has*

$$s_1 + \ldots + s_p = \dim \mathcal{O}_\infty \tag{4.13}$$

Example 4.18. Consider the unicycle described in Example 3.20, whose outputs are the coordinates (x_1, x_2):

$$y_1 = x_1$$

$$y_2 = x_2$$

One has $\mathcal{O}_0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$ and $\mathcal{O}_1 = \mathcal{X}$. Thus, $\sigma_1 = 2$, $\sigma_2 = 1$ and $s_1 = 2$, $s_2 = 1$.

In Chapter 6, we will construct a canonical form that displays the decomposition of a system (1.4) into observable blocks, whose dimensions equal the observability indices.

4.7 Synthesis of Observers

The use of an observer that evaluates the state from the knowledge of inputs and outputs is in order whenever the state itself is not directly measurable, but its value is required for computing a feedback or for monitoring the system behavior. In contrast to the linear situation, observability of a given nonlinear system is necessary but not sufficient to assure the possibility of constructing an observer. In this section, we give some results on the synthesis of a nonlinear observer for a system of the form (1.4), which is based, on one hand, on linearization via output injection and state-space transformation and, on the other hand, on standard Luenberger observer design, performed on the linearized system. The main property which is required for obtaining such an observer is a sort of inherent linearity, which is characterized in the rest of this section. If such a property is absent, it is possible to investigate alternative design techniques, for instance, those giving rise to high gain observers [63] or to sliding mode observers [146], which are not considered here.

4.7.1 Linearization by Input-output Injection and Observer Design 1

Consider a system Σ of the form (1.4) and assume that it is a single output ($p = 1$) and observable system. As a consequence, it has a single observability index which equals n. Assume that it is possible to find a local state-space coordinate transformation $(\xi_1, \ldots, \xi_n) = \phi(x)$ such that

$$\operatorname{rank}\frac{\partial \phi}{\partial x} = n \tag{4.14}$$

and functions $\varphi_i(y,u)$, for $i = 1,\ldots,n$, such that we can write

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \varphi_1(y,u) \\ \quad\vdots \\ \dot{\xi}_{n-1} = \xi_n + \varphi_{n-1}(y,u) \\ \dot{\xi}_n = \varphi_n(y,u) \\ y = \xi_1 \end{cases} \tag{4.15}$$

The terms φ_i in (4.15) define a sort of input-output injection, for which a general definition will be given in Chapter 6. The search for the state-space coordinate transformation $(\xi_1,\ldots,\xi_n) = \phi(x)$ and for the functions $\varphi_i(y,u)$, for $i = 1,\ldots,n$, which together allow us to obtain the form (4.15), is called the linearization problem by input/output injection. The solvability of this latter will be investigated later on. For the moment, starting from (4.15), we show how to construct an observer for Σ.
Note that system (4.15) has the form

$$\begin{cases} \dot{\xi} = A\xi + \varphi(y,u) \\ y = C\xi \end{cases}$$

where (C,A) is a pair of constant matrices in canonical observer form. An estimate $\hat{\xi}$ of the state ξ can then be obtained from the following system:

$$\dot{\hat{\xi}} = A\hat{\xi} + \varphi(y,u) + K(C\hat{\xi} - y) \tag{4.16}$$

where K is chosen so that the eigenvalues of the matrix $A+KC$ are in the open left half complex plane. Thus, the estimation error $\hat{\xi} - \xi$ goes asymptotically to zero. From assumption (4.14), locally $x = \phi^{-1}(\xi_1,\ldots,\xi_n)$ and an estimate $\hat{x}$ for the state x of the original system Σ is given by

$$\hat{x} = \phi^{-1}(\hat{\xi}_1,\ldots,\hat{\xi}_n)$$

Let us go back now to the problem of finding a state-space coordinate transformation $(\xi_1,\ldots,\xi_n) = \phi(x,u)$ and functions $\varphi_i(y,u)$, for $i = 1,\ldots,n$, which allow us to obtain the form (4.15). To find a solution of this problem, if any exists, let us assume that, by applying the state elimination technique of Section 2.1 to a system Σ of form (1.4) and by invoking the implicit function theorem, we get locally an input-output relation of the form

$$y^{(n)} = F(y,\dot{y},\ldots,y^{(n-1)},u,\dot{u},\ldots,u^{(\gamma)}) \tag{4.17}$$

Then, define a sequence of differential one-forms in the following way:

- set $F_0 = F$ and $\varphi_0 = 0$;

- for $k = 1, \ldots, n$, define

$$F_k = F_{k-1} - \varphi_{k-1}^{(n-k+1)} \tag{4.18}$$

$$\omega_k = \frac{\partial F_k}{\partial y^{(n-k)}} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial F_k}{\partial u_j^{(n-k)}} \mathrm{d}u_j \tag{4.19}$$

If $\mathrm{d}\omega_k \neq 0$, stop.
If $\mathrm{d}\omega_k = 0$, then let $\varphi_k(y, u)$ be a solution of

$$\frac{\partial \varphi_k}{\partial y} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial \varphi_k}{\partial u_j} \mathrm{d}u_j = \omega_k \tag{4.20}$$

for $1 \leq k \leq n-1$ and

$$\varphi_n(y, u) = F_n \tag{4.21}$$

At this point, a necessary and sufficient condition for the existence of a state coordinate transformation ϕ which, together with the functions $\varphi_i(y, u)$ constructed above in (4.21), allows us to put the system Σ into the form (4.15), can be formulated as follows:

Theorem 4.19. *Given a single output, observable system Σ of the form (1.4) and letting the function $\varphi_i(y, u)$ for $i = 1, \ldots, n$ be constructed as above in (4.21), there exists locally a state coordinate transformation $\xi = \phi(x)$, satisfying (4.14), such that (4.15) holds if and only if*

$$\mathrm{d}\omega_k = 0 \tag{4.22}$$

for $1 \leq k \leq n$, where the ω_k's are defined by (4.19).

Theorem 4.19 is a special case of Theorem 4.21 below which will be proved in the sequel.

Example 4.20. Let us consider the model of a direct current motor (DC motor), described by the equations (see [18])

$$\begin{cases} \dot{x}_1 = -K_m \cdot x_1 \cdot x_2 - \dfrac{R_a + R_f}{K} \cdot x_1 + u \\ \dot{x}_2 = -\dfrac{B}{J} \cdot x_2 - x_3 + \dfrac{K_m}{J} \cdot K \cdot x_1^2 \\ \dot{x}_3 = 0 \end{cases} \tag{4.23}$$

where x_1 denotes the magnetic flux and verifies $x_1 > 0$; x_2 denotes the rotor speed; x_3 denotes the constant load torque; R_a and R_f denote, respectively, the stator and the inductor resistance; B is the viscous friction coefficient, and K_m is the constant motor torque. As output of the system, to simplify the computations, we choose the natural logarithm of x_1, $y = \ln(x_1)$. Since $\dot{y} = \dfrac{\dot{x}_1}{x_1}$, substituting e^y for x_1, we get

$$\dot{y} = e^{-y}(-K_m e^y x_2 - \frac{R_a + R_f}{K} e^y + u) = -K_m x_2 - \frac{R_a + R_f}{K} + ue^{-y} \quad (4.24)$$

$$y^{(2)} = -K_m(-\frac{B}{J} \cdot x_2 - x_3 + \frac{K_m}{J} \cdot K \cdot e^{2y}) + \dot{u}e^{-y} - ue^{-y}\dot{y} \quad (4.25)$$

$$y^{(3)} = \frac{K_m B}{J}\ \dot{x}_2 - 2\ \frac{K_m^2 K}{J} \cdot e^{2y}\dot{y} + u^{(2)}e^{-y} - 2\dot{u}e^{-y}\dot{y} + ue^{-y}\dot{y}^2 - ue^{-y}y^{(2)} \quad (4.26)$$

From (4.25),

$$(-\frac{B}{J} \cdot x_2 - x_3 + \frac{K_m}{J} \cdot K \cdot e^{2y}) = \frac{1}{K_m}(-y^{(2)} + \dot{u}e^{-y} - ue^{-y}\dot{y}) \quad (4.27)$$

and, by substituting in (4.26), we get the input/output differential equation

$$\begin{aligned} y^{(3)} = F(y, \dot{y}, y^{(2)}, u, \dot{u}, u^{(2)}) &= \frac{B}{J}\ (-y^{(2)} + \dot{u}e^{-y} - ue^{-y}\dot{y}) + \\ -2\ \frac{K_m K}{J} \cdot e^{2y}\dot{y} &+ u^{(2)}e^{-y} - 2\dot{u}e^{-y}\dot{y} + ue^{-y}\dot{y}^2 - ue^{-y}y^{(2)} \end{aligned} \quad (4.28)$$

Let $F_1 = F(y, \dot{y}, y^{(2)}, u, \dot{u}, u^{(2)})$; then $\frac{\partial F_1}{\partial y^{(2)}} = -\frac{B}{J} - ue^{-y}$, $\frac{\partial F_1}{\partial u^{(2)}} = e^{-y}$ and, by (4.19), $\omega_1 = (-\frac{B}{J} - ue^{-y})dy + e^{-y}du$. Now, ω_1 is an exact one-form, since $\omega_1 = d\varphi_1(y, u)$, with $\varphi_1(y, u) = -\frac{B}{J}y + ue^{-y}$.
Following (4.18), define

$$\begin{aligned} F_2(y, \dot{y}, u, \dot{u}) &= F(y, \dot{y}, y^{(2)}, u, \dot{u}, u^{(2)}) - \varphi_1^{(2)}(y, u) \\ &= \frac{B}{J}\ (\dot{u}e^{-y} - ue^{-y}\dot{y}) - 2\ \frac{K_m^2 K}{J}\ e^{2y}\ \dot{y} \end{aligned}$$

Then, $\frac{\partial F_2}{\partial \dot{y}} = -\frac{B}{J}\ u\ e^{-y} - 2\ \frac{K_m^2 K}{J}\ e^{2y}$, $\frac{\partial F_2}{\partial \dot{u}} = \frac{B}{J}\ e^{-y}$
and $\omega_2 = (-\frac{B}{J}\ u\ e^{-y} - 2\ \frac{K_m^2 K}{J}\ e^{2y})dy + (\frac{B}{J}\ e^{-y})du$
Now, ω_2 is an exact one-form because $\omega_2 = d\varphi_2(y, u)$, with

$$\varphi_2(y, u) = \frac{B}{J}\ u\ e^{-y} - \frac{K_m^2 K}{J}\ e^{2y}$$

Define $F_3(y, u) = F_2(y, \dot{y}, u, \dot{u}) - \ \dot{\varphi}_2^{(2)}(y, u) \equiv 0$. Then, $\frac{\partial F_3}{\partial y} \equiv 0$, $\frac{\partial F_3}{\partial u} \equiv 0$, $\omega_3 \equiv 0$, and $\varphi_3(y, u) \equiv 0$.
The change of variables

$$\begin{cases} \xi_1 = \ln(x_1) \\ \xi_2 = -K_m\ x_2 + \frac{B}{J}\ln(x_1) - \frac{R_a + R_f}{K} \\ \xi_3 = K_m\ x_3 - \frac{B}{J} \cdot \frac{R_a + R_f}{K} \end{cases}$$

now yields the canonical form (4.11)

$$\begin{cases} \dot{\xi}_1 = \xi_1 + \varphi_1(y,u) \\ \dot{\xi}_2 = \xi_3 + \varphi_2(y,u) \\ \dot{\xi}_3 = 0 \\ \ y = \xi_1 \end{cases}$$

4.7.2 Linearization by Input-output Injection and Observer Design 2

The problem considered in Section 4.7.1 can be extended by looking for a generalized state-space coordinate transformation $(\xi_1, \dots, \xi_n) = \phi(x, u, \dot{u}, \dots, u^{(s)})$ such that

$$\text{rank}\frac{\partial \phi}{\partial x} = n \tag{4.29}$$

and functions $\varphi_i(y, u, \dot{u}, \dots, u^{(s)})$, for $i = 1, \dots, n$, such that we can write

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \varphi_1(y, u, \dot{u}, \dots, u^{(s)}) \\ \quad \vdots \\ \dot{\xi}_{n-1} = \xi_n + \varphi_{n-1}(y, u, \dot{u}, \dots, u^{(s)}) \\ \dot{\xi}_n = \varphi_n(y, u, \dot{u}, \dots, u^{(s)}) \\ y = \xi_1 \end{cases} \tag{4.30}$$

The terms φ_i in (4.30) are a special form of input-output injection and an observer can be computed from (4.30) as follows.
Note that system (4.30) has the form

$$\begin{cases} \dot{\xi} = A\xi + \varphi(y, u, \dot{u}, \dots, u^{(s)}) \\ y = C\xi \end{cases}$$

where (C, A) is a pair of constant matrices in canonical observer form. Then, an estimate $\hat{\xi}$ of the state ξ is obtained from the following system:

$$\dot{\hat{\xi}} = A\hat{\xi} + \varphi(y, u, \dot{u}, \dots, u^{(s)}) + K(C\hat{\xi} - y) \tag{4.31}$$

where K is chosen so that the eigenvalues of matrix $A + KC$ are in the open left half complex plane. Thus, the estimation error $\hat{\xi} - \xi$ is asymptotically stable. From assumption (4.29), locally $x = \phi^{-1}(\xi_1, \dots, \xi_n, u, \dot{u}, \dots, u^{(s)})$ and an estimate $\hat{x}$ for the original state x of system (1.4) is given by

$$\hat{x} = \phi^{-1}(\hat{\xi}_1, \dots, \hat{\xi}_n, u, \dot{u}, \dots, u^{(s)})$$

Let us go back now to the problem of finding a generalized state-space coordinate transformation $(\xi_1, \dots, \xi_n) = \phi(x, u, \dot{u}, \dots, u^{(s)})$ and a set of functions $\varphi_i(y, u, \dot{u}, \dots, u^{(s)})$, for $i = 1, \dots, n$, which allow us to obtain the form (4.30).
To this aim, let us consider the following sequence of differential one-forms.

- set $F_0 = F$ and $\varphi_0 = 0$;
- for $k = 1, \ldots, n$, define

$$F_k := F_{k-1} - \varphi_{k-1}^{(n-k+1)} \tag{4.32}$$

$$\omega_k = \frac{\partial F_k}{\partial y^{(n-k)}} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial F_k}{\partial u_j^{(n-k+s)}} \mathrm{d}u_j^{(s)} \tag{4.33}$$

Denoting shortly $\wedge \mathrm{d}u_1^{(l)} \wedge \mathrm{d}u_2^{(l)} \wedge \cdots \wedge \mathrm{d}u_m^{(l)}$ by $\wedge \mathrm{d}u^{(l)}$, if $\mathrm{d}\omega_k \wedge \mathrm{d}u \wedge \mathrm{d}\dot{u} \cdots \wedge \mathrm{d}u^{(s-1)} \neq 0$, then stop.
If, otherwise, $\mathrm{d}\omega_k \wedge \mathrm{d}u \wedge \mathrm{d}\dot{u} \cdots \wedge \mathrm{d}u^{(s-1)} = 0$, then let $\varphi_k(y, u, \dot{u}, \cdots, u^{(s)})$ be a solution of

$$\frac{\partial \varphi_k}{\partial y} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial \varphi_k}{\partial u_j^{(s)}} \mathrm{d}u_j^{(s)} = \omega_k \tag{4.34}$$

for $1 \leq k \leq n-1$, and

$$\varphi_n(y, u, \dot{u}, \cdots, u^{(s)}) = F_n \tag{4.35}$$

A necessary and sufficient condition for the existence of a generalized state coordinate transformation $(\xi_1, \ldots, \xi_n) = \phi(x, u, \dot{u}, \ldots, u^{(s)})$ that, together with the functions $\varphi_i(y, u, \dot{u}, \ldots, u^{(s)})$ constructed above, allows us to put the system Σ into the form (4.30), can now be formulated as follows:

Theorem 4.21. *Given a single output, observable system Σ of the form (1.4) and letting the function $\varphi_i(y, u, \dot{u}, \ldots, u^{(s)})$ for $i = 1, \ldots, n$ be constructed as above, there exists locally a state-space coordinate transformation $(\xi_1, \ldots, \xi_n) = \phi(x, u, \dot{u}, \ldots, u^{(s)})$, satisfying (4.29), such that (4.30) holds if and only if*

$$\mathrm{d}\omega_k \wedge \mathrm{d}u \wedge \mathrm{d}\dot{u} \cdots \wedge \mathrm{d}u^{(s-1)} = 0 \tag{4.36}$$

for $1 \leq k \leq n$, where the ω_k's are defined by (4.33).

Before proving Theorem 4.21, let us consider an illustrative example.

Example 4.22. Let Σ be the system defined by the equations

$$\begin{cases} \dot{x}_1 = x_2 \cdot u^2 \\ \dot{x}_2 = 0 \\ \;\; y = x_1 \end{cases} \tag{4.37}$$

The input-output differential equation of Σ is

$$y^{(2)} = F(y, \dot{y}, u, \dot{u}) = \frac{2\dot{y} \cdot \dot{u}}{u} \tag{4.38}$$

For $s = 0$ or $s = 1$, conditions (4.36) are not satisfied, *i.e.*, the linearization problem cannot be solved by a state coordinate transformation and input/injection functions that do not involve time derivatives of u of order greater than 1. Taking $s = 2$ and applying the above procedure, one gets $F_1 = 2\dot{y}\dot{u}/u$. The differential form ω_1 given by (4.33)is

$$\omega_1 = \frac{\partial F_1}{\partial \dot{y}} \mathrm{d}y + \frac{\partial F_1}{\partial u^{(3)}} \mathrm{d}\ddot{u} = \frac{2\dot{u}}{u} \mathrm{d}y \tag{4.39}$$

and condition (4.36) of Theorem 4.21 is satisfied for $k = 1$, since

$$\mathrm{d}\omega_1 \wedge \mathrm{d}u \wedge \mathrm{d}\dot{u} = 0$$

The function $\varphi_1(y, u, \dot{u}, \ddot{u})$ given by (4.34) is

$$\varphi_1(y, u, \dot{u}, \ddot{u}) = \frac{2y \cdot \dot{u}}{u} \tag{4.40}$$

For $k = 2$, $F_2 = 2(\dot{u}^2/u^2 - \ddot{u}/u) \cdot y$, and the differential form ω_2 is

$$\omega_2 = \frac{\partial P_2}{\partial y} \mathrm{d}y + \frac{\partial P_2}{\partial \ddot{u}} \mathrm{d}\ddot{u} = 2(\frac{\dot{u}^2}{u^2} - \frac{\ddot{u}}{u}) \mathrm{d}y - \frac{2y}{u} \mathrm{d}\ddot{u} \tag{4.41}$$

Again, condition (4.36) of Theorem 4.21 is satisfied and the function $\varphi_2(y, u, \dot{u}, \ddot{u})$ given by (4.34) is

$$\varphi_2 = 2(\frac{\dot{u}^2}{u^2} - \frac{\ddot{u}}{u}) \cdot y \tag{4.42}$$

Then, we have the generalized state coordinate transformation $(\xi_1, \xi_2) = (x_1, x_2 u^2 - 2x_1 \cdot \dot{u}/u)$, and we can write

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \dfrac{2y \cdot \dot{u}}{u} \\ \dot{\xi}_2 = 2(\dfrac{\dot{u}^2}{u^2} - \dfrac{\ddot{u}}{u}) \cdot y \\ \;y = \xi_1 \end{cases} \tag{4.43}$$

Comparing with (4.15), an observer for Σ can now be constructed as shown in Section 4.7.1.

Proof (Proof of Theorem 4.21.).
Necessity: Suppose that there exists a generalized state coordinate transformation $\xi = \phi(x, u, \dot{u}, \cdots, u^{(s-1)})$, satisfying (4.29), and functions $\varphi_i(y, u)$ for $i = 1, \ldots, n$ such that (4.30) holds. Then, we can write

$$\begin{aligned} y^{(n)} &= F(\xi, u, \dot{u}, \cdots, u^{(s-1)}) \\ &= \varphi_1^{(n-1)} + \varphi_2^{(n-2)} + \cdots + \varphi_n \\ &= F_0 \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_1 &= F_0 \\ &= \frac{\partial \varphi_1}{\partial y} y^{(n-1)} + \sum_{j=1}^{m} \frac{\partial \varphi_1}{\partial u_j^{(s)}} u_j^{(n-1+s)} + \Theta_1(y, \cdots, y^{(n-2)}, u, \cdots, u^{(n-2+s)}) \end{aligned}$$

and the differential form ω_1, according to (4.33), is given by

$$\omega_1 = \frac{\partial F_1}{\partial y^{(n-1)}} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial F_1}{\partial u_j^{(n-1+s)}} \mathrm{d}u_j^{(s)}$$
$$= \frac{\partial \varphi_1}{\partial y} \mathrm{d}y + \sum_{j=1}^{m} \frac{\partial \varphi_1}{\partial u_j^{(s)}} \mathrm{d}u_j^{(s)}$$

Then, the condition $\mathrm{d}\omega_1 \wedge \mathrm{d}u \wedge \mathrm{d}\dot{u} \wedge \cdots \wedge \mathrm{d}u^{(s-1)} = 0$ is satisfied. The proof for steps $2 \le k \le n$ follows the same lines.
Sufficiency: Assume that the condition (4.36) is satisfied and let $\varphi_k(y, u, \dot{u}, \cdots, u^{(s)})$ be given by (4.34). From the definition of $\varphi_1, \ldots, \varphi_{n-1}$,

$$\dot{\xi}_i = \xi_{i+1} + \varphi_i, \text{ for } i = 1, \ldots, n-1$$

Computing $\dot{\xi}_n$, one gets

$$\dot{\xi}_n = y^{(n)} - \varphi_{n-1}^{(1)} - \varphi_{n-2}^{(2)} - \cdots - \varphi_1^{(n-1)} \tag{4.44}$$

and, finally, from (4.36), (4.32), and (4.34), $y^{(n)} = \varphi_n + \varphi_{n-1}^{(1)} + \cdots + \varphi_1^{(n-1)}$. Therefore, $\dot{\xi}_n = \varphi_n$, and the result follows. ∎

Problems

4.1. Given a system Σ of the form (1.4), show that the equality $\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$ holds.

4.2. Given a system Σ of the form (1.4), prove that

$$\mathit{dim}\ \mathcal{O}_\infty = \mathrm{rank}_{\mathcal{K}} \left[\frac{\partial(y, \dot{y}, \ldots, y^{(n-1)})}{\partial x} \right] \tag{4.45}$$

4.3. Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Compute the spaces $\mathcal{Y}^k$ and $\mathcal{O}_k$, for $k \ge 1$, in terms of the matrices A and C, and derive the standard observability criterion for linear systems.

5

Systems Structure and Inversion

Given a system Σ with input u and output y, the idea of designing an inverse system Σ^{-1}, whenever it is possible, is quite appealing, since the inverse system may provide a way to compute the control input $u(t)$ that is required to obtain a (desired) output $y(t)$ from Σ. The notion of inverse system is introduced and discussed in this chapter, as well as that of an inversion algorithm. Minimality of inverse systems is considered as well and it is discussed in the light of the intrinsic notion of zero dynamics. This, in turn, is shown to play a key role in stabilization problems and in output tracking problems with internal stability.

5.1 Introductory Examples

5.1.1 A Resistor Circuit

Let Σ be a static system (*i.e.* consisting only of algebraic equations), whose input u and output y represent, respectively, the voltage and the current in a resistor R.

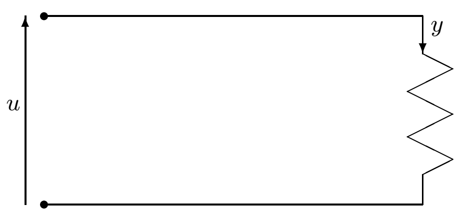

Fig. 5.1. Resistor circuit

The input-output equation of Σ is

$$y = (1/R)u$$

Its inverse system Σ^{-1} operates generating its "output" $u(t)$ from the "input" $y(t)$ and it is easily defined through its input-output equation

$$u = Ry$$

Clearly, one can use Σ^{-1} to determine the input required to force the system Σ to produce any desired output.

5.1.2 An Induction-resistor Circuit

Now let Σ denote the series connection of an induction element L with a resistor R. The input $u(t)$ represents the voltage and the output $y(t)$ represents the current in the circuit. The input-output equation of Σ is $\dot{y} = -(R/L)y + u$ and a state-space realization is

$$\begin{cases} \dot{x} = -(R/L)x + u \\ y = x \end{cases}$$

An inverse system Σ^{-1} should take $y(t)$ as input and it should produce $u(t)$ as output. A candidate inverse system can be described by the set of equations

$$\begin{cases} \dot{z} = \dot{y} \\ u = (R/L)z + \dot{y} \\ z(0) = x(0) \end{cases}$$

A different candidate, however, may be described in a simpler way by the equation

$$u = \dot{y} + (R/L)y$$

5.2 Inverse Systems

To discuss the notion of system inversion in general terms, let us consider a system Σ of the form (1.4), that is,

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.1}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $y \in I\!R^p$, and the entries of f, g and h are meromorphic functions of x.

Definition 5.1. Right inverse *The system*

$$\begin{cases} \dot{z} = F(z, y, \dot{y}, \ldots, y^{(\nu)}) \\ u = H(z, y, \dot{y}, \ldots, y^{(\nu)}) \end{cases} \tag{5.2}$$

is a right inverse system for Σ if there exists $z(0)$ such that the output $y(t)$ of (5.1) equals the input $y(t)$ of (5.2) whenever the input $u(t)$ of (5.1) is chosen as the output of (5.2).

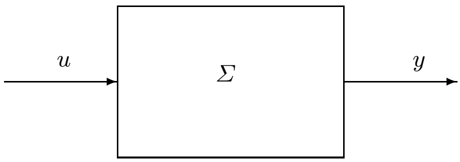

Fig. 5.2. System Σ

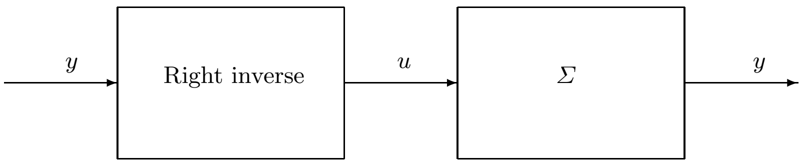

Fig. 5.3. Right inverse system

The definition of a left inverse is obtained essentially by interchanging the roles of (5.1) and (5.2).

Definition 5.2. Left inverse *The system*

$$\begin{cases} \dot{z} = F(z, y, \dot{y}, \ldots, y^{(\nu)}) \\ u = H(z, y, \dot{y}, \ldots, y^{(\nu)}) \end{cases} \tag{5.3}$$

is a left inverse system for system (5.1) if the output $u(t)$ of (5.3) equals the input $u(t)$ of (5.1) whenever the input $y(t)$ of (5.3) is chosen as the output of (5.1).

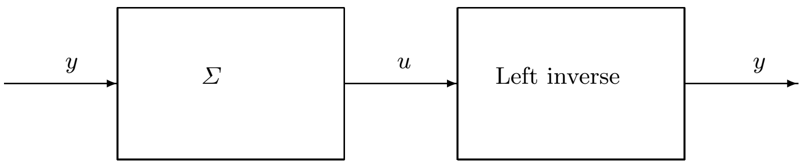

Fig. 5.4. Left inverse system

5.3 Structural Indices

To handle the problem of constructing, if possible, the inverse of a given system, we need to introduce a number of tools and notions related to structural properties. To begin with, let us recall that our study of accessibility was based on the notion of relative degree, given in Definition 3.7, of a function, that may be viewed as an output. In this case, the relative degree represents the *delay* existing between the control input and the output function. More precisely, it is the order of differentiation which has to be applied to the output to have explicit dependence on the input. From this point of view, this notion describes the so-called structure at infinity in the single output case. More generally, the structure at infinity of a nonlinear system displays, roughly speaking, the delay structure existing between the input and the output in the multivariable case. In this section, the structure at infinity of a nonlinear system will be formally defined and studied, and an algorithm for computing it will be introduced.

5.3.1 Structure at Infinity

Given the system (5.1), one can naturally associate with Σ the chain of subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n$ of $\mathcal{E}$ defined by

$$\begin{aligned} \mathcal{E}_0 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \\ &\vdots \\ \mathcal{E}_n &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(n)}\} \end{aligned} \tag{5.4}$$

Definition 5.3. *Given the chain of vector spaces* (5.4)*, the list of integers* $\{\sigma_k, k = 1, \ldots, n\}$ *defined by*

$$\sigma_k = \dim_{\mathcal{K}} \frac{\mathcal{E}_k}{\mathcal{E}_{k-1}} \tag{5.5}$$

is called the **structure at infinity of** Σ.

The list given by (5.5) contains structural information on the system that plays a crucial role in the solution of many control problems ([22, 42, 132]). The list $\{s_k, k = 1, \ldots, n\}$, defined by

$$s_1 = \sigma_1, \;\; s_k = \sigma_k - \sigma_{k-1}, \; k = 2, \ldots, n \tag{5.6}$$

describes the so-called zeros at infinity of Σ as follows:

- s_1 is the number of zeros at infinity whose order equals 1;
- s_i is the number of zeros at infinity whose order equals i.

In relation to the structure at infinity, two more lists of integers can be considered.

- $\{n_1, \ldots, n_p\}$, where n_i is the the relative degree of the output component y_i; each n_i is said to represent the order of the zero at infinity of the output component with which it is associated;

- $\{n'_1, \ldots, n'_p\}$, the list of integers obtained by duality from $\{s_k, k = 1, \ldots, n\}$ (*i.e.*, from the relation $\sigma_i = \mathrm{card}\{n'_j \mid n'_j \leq i\}$); such a list is said to represent the orders of the zeros at infinity of the system.

One more notion will be used in the following.

Definition 5.4. *The essential order n_{ie} of the scalar output component y_i is defined by*

$$\begin{aligned} n_{ie} &= \min\{k \in I\!N \mid \\ &\quad \mathrm{d}y_i^{(k)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}, \mathrm{d}y_{j\neq i}^{(k)}, \mathrm{d}y^{(k+1)}, \ldots, \mathrm{d}y^{(n)}\}\}. \end{aligned} \tag{5.7}$$

To make the meaning of these various lists of integers clearer, let us consider the following example.

Example 5.5. Let Σ be the system defined by the equations

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_3 u_1 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

Computing the derivatives of the output, we get

$$\begin{cases} \dot{y}_1 = u_1 \\ \dot{y}_2 = x_3 u_1 = x_3 \dot{y}_1 \end{cases}$$

Then, $\{n_1,\ n_2\} = \{1,\ 1\}$, since a component of the input appears in the first derivative of both output components. However, to get all the components of the input, we have to derive $\dot{y}_2$ further.

$$\ddot{y}_2 = \dot{x}_3 \dot{y}_1 + x_3 \ddot{y}_1 = u_2 \dot{y}_1 + x_3 \ddot{y}_1$$

Then, $\{n'_1,\ n'_2\} = \{1,\ 2\}$. Now, let us compute the subspaces (5.4) associated with the system Σ.

$$\mathcal{E}_0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}, \quad \mathcal{E}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_1\}, \quad \mathcal{E}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_1, \mathrm{d}\dot{u}_1, \mathrm{d}u_2\}$$

Hence, $\sigma_1 = 1$ and $\sigma_2 = 2$. We have also that $n_{1e} = 2$ and $n_{2e} = 2$.

Note that for a single-output system the relative degree of $y = h(x)$ equals the order of the zero at infinity of Σ.
This fact motivates the following definition.

Definition 5.6. *The relative degree of a single output system Σ is the relative degree of its output function.*

A general algorithm for computing the structure at infinity of a given system is discussed in Section 5.4.

5.4 Structure Algorithm

The structure at infinity of a given system can be computed by applying a suitable algorithm, called the *structure algorithm*, that turns out to be a fundamental tool in the analysis of nonlinear dynamic systems. The structure algorithm was first introduced by Silverman for linear time-invariant systems [144], and it was then generalized to nonlinear systems in [74]. Then, Singh further extended it in [145] and finally it appeared in [42] in the form we will present hereafter.
The structure algorithm is also known as the *inversion algorithm*, or Singh's inversion algorithm, since, when a system is invertible, it allows us to express the input as a function of the output, its time derivatives and, possibly, some states. Thus it may be viewed as an algorithm that directly computes the input necessary for generating a desired output function.
Control methods such as trajectory tracking or computed torque control in robotics are special applications of the inversion algorithm.
To describe the algorithm, assume that a system of the form (5.1) is given.

Algorithm 5.7 (The Structure Algorithm)

Step 1.
Compute

$$\dot{y} = \frac{\partial h(x)}{\partial x}(f(x) + g(x)u)$$

and write

$$a_1(x) := \frac{\partial h(x)}{\partial x} f(x)$$

$$b_1(x) := \frac{\partial h(x)}{\partial x} g(x)$$

Then, $\dot{y} = a_1(x) + b_1(x)u$*. Define*

$$\rho_1 = \operatorname{rank} b_1(x)$$

Permute, if necessary, the components of the output, so that the first ρ_1 *rows of* $b_1(x)$ *are linearly independent over* $\mathcal{K}$*. Denote by* $\dot{\tilde{y}}_1$ *the vector consisting of the first* ρ_1 *rows of* $\dot{y}$*, and denote by* $\dot{\hat{y}}_1$ *the other* $p - \rho_1$ *rows, so that*

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix}$$

Since the last rows of $b_1(x)$ *are linearly dependent upon the first* ρ_1 *rows, we can write*

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \dot{\hat{y}}_1 &= \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) \end{aligned}$$

where the last equation is affine in $\dot{\tilde{y}}_1$*. Finally, define*

$$\tilde{B}_1(x) := \tilde{b}_1(x)$$

Step k+1.

Suppose that in Steps 1 through k, $\dot{\tilde{y}}_1, \dots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ *have been defined so that*

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ &\vdots \\ \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_1^{(k)}, \dots, \tilde{y}_{k-1}^{(k-1)}, \tilde{y}_{k-1}^{(k)}) \\ &\quad + \tilde{b}_k(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_1^{(k-1)}, \dots, \tilde{y}_{k-1}^{(k-1)})u \end{aligned} \tag{5.8}$$

$$\hat{y}_k^{(k)} = \hat{y}_k^{(k)}(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_1^{(k)}, \dots, \tilde{y}_k^{(k)})$$

For every i, j, *and* k, $\tilde{y}_i^{(j)}$ *and* $\hat{y}_k^{(k)}$ *are meromorphic functions in* $\mathcal{K}$.
Suppose that the matrix $\tilde{B}_k := [\tilde{b}_1^T, \cdots, \tilde{b}_k^T]^T$ *has a full row rank equal to* ρ_k. *Then, compute*

$$\hat{y}_k^{(k+1)} = \frac{\partial \hat{y}_k^{(k)}}{\partial x}[f(x) + g(x)u] + \sum_{i=1}^{k}\sum_{j=i}^{k} \frac{\partial \hat{y}_k^{(k)}}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)}$$

and write it as

$$\begin{aligned} \hat{y}_k^{(k+1)} &= a_{k+1}(x, \tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k+1) \\ &\quad + b_{k+1}(x, \tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k)\ u \end{aligned}$$

Define $B_{k+1} := [\tilde{B}_k^T, b_{k+1}^T]^T$, *and*

$$\rho_{k+1} := \text{rank } B_{k+1}$$

Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ *so that the first* ρ_{k+1} *rows of* B_{k+1} *are linearly independent. Decompose* $\hat{y}_k^{(k+1)}$ *as*

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix}$$

where $\tilde{y}_{k+1}^{(k+1)}$ *consists of the first* $(\rho_{k+1} - \rho_k)$ *rows. Since the last rows of* B_{k+1} *are linearly dependent on the first* ρ_{k+1} *rows, one can write*

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ &\vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k+1) \\ &\quad + \tilde{b}_{k+1}(x, \tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k)u \end{aligned} \tag{5.9}$$

$$\hat{y}_{k+1}^{(k+1)} = \hat{y}_{k+1}^{(k+1)}(x, \tilde{y}_i^{(j)}, 1 \le i \le k+1, i \le j \le k+1). \tag{5.10}$$

Finally, set $\tilde{B}_{k+1} := [\tilde{B}_k^T, \tilde{b}_{k+1}^T]^T$.
The algorithm stops when

$$\mathrm{rank}[\partial(y, \dot{\hat{y}}_1, \ldots, \hat{y}_{k+1}^{(k+1)})]/\partial x = \mathrm{rank}[\partial(y, \dot{\hat{y}}_1, \ldots, \hat{y}_k^{(k)})]/\partial x.$$

End of the algorithm.

Assuming that Algorithm 5.7 stops at step k, the orders of the zeros at infinity, as well as the essential orders, may be computed from (5.9). The list of the orders of the zeros at infinity is given by the list of the smallest orders of differentiation of $y_i, i = 1, \ldots, p$ that occur in (5.9), and the essential order n_{ie} equals the largest order of differentiation of y_i in (5.9).
Actually, the structure algorithm computes a basis for the chain of subspaces (5.4).

Theorem 5.8. *[42] A basis for* $\mathcal{E}_k$ *is given by*

$$\{\mathrm{d}x, \mathrm{d}\dot{\tilde{y}}_1, \ldots, \mathrm{d}\tilde{y}_1^{(k)}, \ldots, \mathrm{d}\tilde{y}_{k-1}^{(k-1)}, \mathrm{d}\tilde{y}_{k-1}^{(k)}, \mathrm{d}\tilde{y}_k^{(k)}\}$$

Proof. Note that for $k = 1$, $\{\mathrm{d}x, \mathrm{d}\dot{\tilde{y}}_1\}$ is a set of generators of $\mathcal{E}_1$ since $\mathrm{d}\dot{\hat{y}}_1 \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\dot{\tilde{y}}_1\}$. Since $\tilde{b}_1(x)$ has a full row rank, the set also consists of independent vectors and it is thus a basis for $\mathcal{E}_1$. The rest of the proof may be done by induction and is reported in [42]. ∎

It follows from the above theorem that the two lists $\{\sigma_k, k = 1, \ldots, n\}$ and $\{\rho_k, k = 1, \ldots, n\}$ coincide.

Definition 5.9. *The integer* ρ_n *obtained by Algorithm 5.7 is called the rank of the system* Σ.

5.4.1 A Pseudoinverse System

A so-called pseudoinverse system Σ^{-1} can now be constructed by Algorithm 5.7. Assume that the algorithm stops at Step k, then the matrix $\tilde{B}_k$ has a full rank equal to ρ, and *rank* $B_k =$ *rank* $B_{k+1} = \cdots =$ *rank* B_n.
Then, the first ρ components of u, say $(u_1, \cdots, u_\rho)$, can be obtained by solving the system (5.9), and we can write

$$\begin{bmatrix} u_1 \\ \vdots \\ u_\rho \end{bmatrix} = F(x, \dot{y}, \cdots, y^{(n)}, u_{\rho+1}, \cdots, u_m) \tag{5.11}$$

for some suitable function F.
The system Σ^{-1} will be defined as follows.

$$\Sigma^{-1}: \begin{cases} \dot{z} & = f(z) + g(z) \begin{bmatrix} F(z, \dot{y}, \cdots, y^{(n)}, u_{\rho+1}, \cdots, u_m) \\ u_{\rho+1} \\ \vdots \\ u_m \end{bmatrix} \\ \begin{bmatrix} u_1 \\ \vdots \\ u_\rho \end{bmatrix} & = F(z, \dot{y}, \cdots, y^{(n)}, u_{\rho+1}, \cdots, u_m) \end{cases} \tag{5.12}$$

In the special case of square invertible systems *(i.e.* $\rho_k = m = p$*)*, the system (5.9) has a unique solution u; therefore, an inverse system Σ^{-1} is uniquely defined.

5.4.2 Examples

Example 5.10. Consider the unicycle in Example 4.18. The inversion algorithm 5.7 yields

$$\dot{y}_1 = (\cos\, x_3) u_1 \tag{5.13}$$

$$\dot{y}_2 = (\sin\, x_3) u_1 = (\tan\, x_3) \dot{y}_1 \tag{5.14}$$

The second step of Algorithm 5.7 gives

$$\ddot{y}_2 = (\tan\, x_3) \ddot{y}_1 + (1/\cos^2\, x_3) \dot{y}_1 u_2$$

and the algorithm stops. The input is then obtained as

$$\begin{aligned} u_1 &= \dot{y}_1 / \cos\, x_3 \\ u_2 &= \frac{[\ddot{y}_2 - (\tan\, x_3)\ddot{y}_1] \cos^2\, x_3}{\dot{y}_1} \end{aligned} \tag{5.15}$$

A pseudoinverse is thus obtained as

$$\Sigma_1^{-1}: \left[\begin{aligned} \dot{z}_1 &= \dot{y}_1 \\ \dot{z}_2 &= (\tan z_3) \dot{y}_1 \\ \dot{z}_3 &= \frac{[\ddot{y}_2 - (\tan\, z_3)\ddot{y}_1] \cos^2\, z_3}{\dot{y}_1} \\ u_1 &= \dot{y}_1 / \cos\, z_3 \\ u_2 &= \frac{[\ddot{y}_2 - (\tan\, z_3)\ddot{y}_1] \cos^2\, z_3}{\dot{y}_1} \end{aligned} \right. \tag{5.16}$$

Step 1 of the Structure Algorithm may be performed in a different way, writing

$$\dot{y}_2 = (\sin\, x_3) u_1 \tag{5.17}$$

$$\dot{y}_1 = (\sin\, x_3) u_1 = (\cot\, x_3) \dot{y}_2 \tag{5.18}$$

A different pseudoinverse is then obtained as

$$\Sigma_2^{-1}: \left[\begin{array}{l} \dot{\zeta}_1 = \dot{y}_2 \cot \zeta_3 \\ \dot{\zeta}_2 = \dot{y}_2 \\ \dot{\zeta}_3 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2 \zeta_3}{\dot{y}_2} \\ u_1 = \dot{y}_2 / \sin \zeta_3 \\ u_2 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2 \zeta_3}{\dot{y}_2} \end{array} \right. \tag{5.19}$$

The space $\mathcal{E}_1$ admits the basis $\{\mathrm{d}x, \mathrm{d}\dot{y}_1\}$ and the space $\mathcal{E}_2$ admits the basis $\{\mathrm{d}x, \mathrm{d}\dot{y}_1, \mathrm{d}\ddot{y}_1, \mathrm{d}\ddot{y}_2\}$. The unicycle has one zero at infinity of order 1 and one zero at infinity of order 2. Both essential orders equal 2.

Example 5.11. Consider the following system with two inputs and one output.

$$\begin{cases} \dot{x}_1 = x_2 u_1 + u_2 \\ \dot{x}_2 = u_2 \\ \ \ y = x_1 \end{cases} \tag{5.20}$$

The Structure Algorithm yields $\dot{y} = x_2 u_1 + u_2$ which can be solved either in u_1 or in u_2. In one case, one gets the pseudoinverse Σ_1^{-1}

$$\Sigma_1^{-1}: \left[\begin{array}{l} \dot{z}_1 = \dot{y} \\ \dot{z}_2 = \dot{y} - z_2 u_1 \\ u_2 = \dot{y} - z_2 u_1 \end{array} \right. \tag{5.21}$$

and in the second case, the pseudoinverse Σ_2^{-1} is obtained

$$\Sigma_2^{-1}: \left[\begin{array}{l} \dot{\zeta}_1 = \dot{y} \\ \dot{\zeta}_2 = u_2 \\ u_1 = \dfrac{\dot{y} - u_2}{\zeta_2} \end{array} \right. \tag{5.22}$$

Different (pseudo-)inverse systems can be obtained by choosing different components for $\tilde{y}_k^{(k)}$ in the Structure Algorithm 5.7. This can be seen also in the case of the unicycle in Example 4.18.
Step 1 of the Structure Algorithm may be performed in a different way, writing

$$\dot{y}_2 = (\sin x_3) u_1 \tag{5.23}$$

$$\dot{y}_1 = (\sin x_3) u_1 = (\cot x_3) \dot{y}_2 \tag{5.24}$$

Thus, a pseudoinverse, different from (5.16), is obtained:

$$\Sigma_2^{-1}: \left[\begin{array}{l} \dot{\zeta}_1 = \dot{y}_2 \cot \zeta_3 \\ \dot{\zeta}_2 = \dot{y}_2 \\ \dot{\zeta}_3 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2 \zeta_3}{\dot{y}_2} \\ u_1 = \dot{y}_2 / \sin \zeta_3 \\ u_2 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2 \zeta_3}{\dot{y}_2} \end{array} \right. \tag{5.25}$$

Example 5.12. Let us continue with the example of the unicycle. Both in (5.16) and in (5.19), the output of the inverse system depends only on the third dynamic. Thus,

$$\Sigma_{1r}^{-1} =: \left[\begin{array}{l} \dot{z}_3 = \dfrac{[\ddot{y}_2 - (\tan\, z_3)\ddot{y}_1] \cos^2\, z_3}{\dot{y}_1} \\ u_1 = \dot{y}_1 / \cos\, z_3 \\ u_2 = \dfrac{[\ddot{y}_2 - (\tan\, z_3)\ddot{y}_1] \cos^2\, z_3}{\dot{y}_1} \end{array} \right. \tag{5.26}$$

is a (reduced) inverse system as well as

$$\Sigma_{2r}^{-1} : \left[\begin{array}{l} \dot{\zeta}_3 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2\, \zeta_3}{\dot{y}_2} \\ u_1 = \dot{y}_2 / \sin\, \zeta_3 \\ u_2 = \dfrac{[\ddot{y}_2 \cot \zeta_3 - \ddot{y}_1] \sin^2\, \zeta_3}{\dot{y}_2} \end{array} \right. \tag{5.27}$$

Moreover, the relation $z_3 = \arctan(\dot{y}_2/\dot{y}_1)$ obtained from (5.14), where z_3 has been substituted for x_3, allows us to obtain a static (dynamic-free) inverse:

$$\begin{cases} u_1 = \sqrt{\dot{y}_1^2 + \dot{y}_2^2} \\ u_2 = \dfrac{[\ddot{y}_2 \dot{y}_1 - \ddot{y}_1 \dot{y}_2]}{\dot{y}_1^2 + \dot{y}_2^2} \end{cases}$$

The same result can be obtained by eliminating ζ_3 in (5.27).

Example 5.13. Consider again the circuit of Section 5.1.2

$$\begin{cases} \dot{x} = -(R/L)x + u \\ y = x \end{cases}$$

The Structure Algorithm provides $\dot{y} = -(R/L)x + u$ and the inverse system

$$\begin{cases} \quad \dot{z} = \dot{y} \\ \quad u = (R/L)z + \dot{y} \\ z(0) = x(0) \end{cases}$$

However, since we have $y = x$ from the system's equations, a static, state-free inverse system can be obtained as

$$u = \dot{y} + (R/L)y$$

5.4.3 Reduced Inverse Systems

From the last Example, it is clear that the complexity of inverse systems obtained by (5.12) may be reduced [165]. In this regard, we have the following result. Let us denote by H the matrix

$$H(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)} := \begin{bmatrix} h(x) \\ \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{y}_k^{(k)}(x, \dot{\tilde{y}}_1, \ldots, \tilde{y}_1^{(k)}, \ldots, \tilde{y}_k^{(k)}) \end{bmatrix} \tag{5.28}$$

whose rows are given by the output equations of system (5.1) and from the inversion equations (5.9).

Theorem 5.14. *Given the system Σ, there exists a pseudoinverse of dimension*

$$s = n - \text{rank}\frac{\partial H}{\partial x}$$

Proof. Consider the set of equations

$$H(x, \dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}) - \hat{y} = 0 \tag{5.29}$$

where H is the matrix (5.28) and

$$\hat{y} := \begin{bmatrix} y \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}_k^{(k)} \end{bmatrix}$$

Substitute z_i for x_i for $i = 1, ..., n$ in H. Without loss of generality, assume that the $n-s$ first rows of $\partial H/\partial z$ are independent and denote by $\tilde{H}$ the $(n-s)$ dimensional vector consisting of the first $(n-s)$ components of $H(z, \dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)})$. Without loss of generality, we can also assume that

$$\text{rank}\frac{\partial \tilde{H}}{\partial(z_1, ..., z_{n-s})} = n - s$$

By the implicit function theorem, we can solve in $(z_1, ..., z_{n-s})$ the equation

$$\tilde{H}(z, \dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}) - \bar{y} = 0$$

where $\bar{y}$ is the vector consisting of the $(n-s)$ first components of $\hat{y}$. Thus,

$$\begin{pmatrix} z_1 \\ \vdots \\ z_{n-s} \end{pmatrix} = \hat{H}(z_{n-s+1}, ..., z_n, \dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}) \tag{5.30}$$

for some suitable function $\hat{H}$. Rewrite (5.12), removing the $n-s$ first equations of the dynamic $\dot{z}$. In $F(z_1, ..., z_{n-s}, zn - s + 1, ..., z_n, \dot{y}, \cdots, y^{(n)}, u_{\rho+1}, \cdots, u_m)$, substitute $(z_1, ..., z_{n-s})$ using (5.30) to get the reduced inverse system Σ_R^{-1} defined by the equations

$$\begin{cases} \begin{pmatrix} \dot{z}_{n-s+1} \\ \vdots \\ \dot{z}_n \end{pmatrix} = \tilde{f}(\cdot) + \tilde{g}(\cdot) \begin{bmatrix} \tilde{F}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}, u_{\rho+1}, \cdots, u_m) \\ u_{\rho+1} \\ \vdots \\ u_m \end{bmatrix} \\ \begin{bmatrix} u_1 \\ \vdots \\ u_\rho \end{bmatrix} = \tilde{F}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}, u_{\rho+1}, \cdots, u_m) \end{cases} \tag{5.31}$$

where $\tilde{f}(\cdot) = \tilde{f}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y})$ and $\tilde{g}(\cdot) = \tilde{g}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y})$. Σ_R^{-1} is the desired reduced inverse of dimension s. ■

5.5 Invertibility

We can now state the main results about the existence of inverse systems.

Definition 5.15. *System (1.1) is said to be left- (resp., right-) invertible if there exists a left (resp., right) inverse system according to Definitions 5.2 and 5.1 respectively.*

Proposition 5.16. *[51] System (1.1) is left- (resp., right-) invertible if and only if its rank ρ equals m (resp., p).*

Example 5.17. Example 5.10 continued.
From (5.15), a left inverse system of the unicycle is given by the equations

$$\begin{cases} \dot{\eta} = \dfrac{[\ddot{y}_2 - (\tan\eta)\ddot{y}_1]\cos^2\eta}{\dot{y}_1} \\ u_1 = \dot{y}_1/\cos\eta \\ u_2 = \dfrac{[\ddot{y}_2 - (\tan\eta)\ddot{y}_1]\cos^2\eta}{\dot{y}_1} \end{cases}$$

From (5.14), $x_3 = \arctan(\dot{y}_2/\dot{y}_1)$ and thus, replacing x_3 by η, a static inverse system is given by the equations

$$\begin{cases} u_1 = \sqrt{\dot{y}_1^2 + \dot{y}_2^2} \\ u_2 = \dfrac{\dot{y}_1\ddot{y}_2 - \dot{y}_2\ddot{y}_1}{\dot{y}_1^2 + \dot{y}_2^2} \end{cases}$$

Example 5.18. Consider the linear system whose transfer function equals $(s+1)/s^2$ and in state-space form reads

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 + x_2 \end{cases}$$

The inversion algorithm yields $\dot{y} = x_2 + u$ and thus, a "reduced" inverse [165] is

$$\begin{cases} \dot{\eta} = -\eta + \dot{y} \\ u = -\eta + \dot{y} \end{cases}.$$

In the linear case, the existence of a reduced inverse system, as it happens in Example 5.18, is explained by the presence of one transmission zero. In the non linear case, the situation is more complicated but, nevertheless, interesting relations have been established.

5.6 Zero Dynamics

In the nonlinear context, there exist three different notions that generalize the concept of transmission zeros of a linear time-invariant system, as it is shown in [88]. A fundamental one is the notion of zero dynamics that can be introduced by reduced inverse dynamics, as defined in the previous section.
To clarify the situation, we start by a particular case, under the following assumptions.

Assumption 5.19 *System (5.1) is minimal, i.e., observable and accessible.*

Assumption 5.20 *System (5.1) is square invertible.*

Assumption 5.21 *The trajectory $y \equiv 0$ is not singular for the reduced inverse*

$$\Sigma_R^{-1} : \begin{cases} \begin{pmatrix} \dot{z}_{n-s+1} \\ \vdots \\ \dot{z}_n \end{pmatrix} = \tilde{f}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}) \\ \qquad + \tilde{g}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}) \left[\tilde{F}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}) \right] \\ \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \tilde{F}(z_{n-s+1}, ..., z_n, \tilde{y}, \hat{y}) \end{cases} \tag{5.32}$$

The dynamics

$$\begin{pmatrix} \dot{z}_{n-s+1} \\ \vdots \\ \dot{z}_n \end{pmatrix} = \tilde{f}(z_{n-s+1}, ..., z_n, 0, ..., 0) \\ + \tilde{g}(z_{n-s+1}, ..., z_n, 0, ..., 0)\tilde{F}(z_{n-s+1}, ..., z_n, 0, ..., 0) \tag{5.33}$$

are well defined.

Definition 5.22. *The zero dynamics of system (5.1) is defined by the dynamics (5.33) of a reduced inverse driven by $y \equiv 0$.*

By the structure algorithm, we can immediately obtain the following:

Proposition 5.23. *If the system is invertible, then the dimension of the zero dynamics equals*

- $n - \sum_{i=1}^{i=m} n'_i$, *where the* n'_i *denote the orders of the zeros at infinity,*
- $\dim \mathcal{O}_\infty - \dim(\mathcal{X} \cap \mathcal{Y})$.

Examples

Example 5.24. Given the linear system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + x_2 + u \\ \dot{x}_3 = 0 \\ \ \ y = x_1 + x_3 \end{cases}$$

whose transfer function is $\dfrac{1}{s^2 - s - 1}$, the inversion algorithm gives

$$\dot{y} = x_2, \qquad y^{(2)} = x_1 + x_2 + u \tag{5.34}$$

Because $\rho = 2$, the system is invertible and the inverse system is given by the equations

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = y^{(2)} \\ \dot{z}_3 = 0 \\ \ \ u = y^{(2)} - z_1 - z_2 \end{cases} .$$

Equations (5.29) are, in this case, $y = x_1 + x_3$ and $\dot{y} = x_2$. Solving with respect to x_1 and x_2 and substituting z_i for x_i, for $i = 1, 2$, we obtain

$$\begin{cases} \dot{z}_1 = \dot{y} \\ \dot{z}_2 = y^{(2)} \\ \dot{z}_3 = 0 \\ \ \ u = y^{(2)} - \dot{y} - y + z_3 \end{cases}$$

The reduced inverse is therefore

$$\begin{cases} \dot{z}_3 = 0 \\ \ \ u = y^{(2)} - \dot{y} - y + z_3 \end{cases}$$

whose transfer function is $s^2 - s - 1$.
Note that this is an inverse system in the sense of Definition 5.2, since the substitution of y and $\dot{y}$ in $u = y^{(2)} - \dot{y} - y + z_3$ yields $u = u - x_3 + z_3$. Pick $z_3(0) = x_3(0)$ and the result follows.
A minimal realization of the transfer function $s^2 - s - 1$ consists of the static system

$$u = y^{(2)} - \dot{y} - y$$

which is not an inverse system in the sense of Definition 5.2 since the substitution of (5.34) in $u = y^{(2)} - \dot{y} - y$ yields $u = u - x_3$.

Example 5.25. Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -x_1^2 - u^\alpha \\ \ \ y = x_1 \end{cases}$$

The inverse system is

$$\begin{cases} \dot{z}_1 = \dot{y} \\ \dot{z}_2 = -z_1^2 - (\dot{y} - z_2)^\alpha \\ \ \ u = \dot{y} - z_2 \end{cases}$$

Substitute y for z_1 and get the reduced inverse system

$$\begin{cases} \dot{z}_2 = -y^2 - (\dot{y} - z_2)^\alpha \\ \ \ u = \dot{y} - z_2 \end{cases}$$

Set $y \equiv 0$, and the zero dynamics is obtained as

$$\dot{z}_2 = (-1)^{\alpha+1} z_2^\alpha$$

Example 5.26.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2 + u \\ \dot{x}_3 &= 0 \\ y &= x_1 \cdot x_3 \end{aligned}$$

The inversion algorithm yields

$$\begin{array}{lll} a) & y & = x_1 \cdot x_3 \\ b) & \dot{y} & = x_2 \cdot x_3 \\ c) & y^{(2)} & = (x_1 + x_2)x_3 + ux_3 \end{array} \tag{5.35}$$

For $x_3 \neq 0$, the inverse system is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_1 + z_2 + \frac{y^{(2)}}{z_3} - z_1 - z_2 = \frac{y^{(2)}}{z_3} \\ \dot{z}_3 &= 0 \\ u &= \frac{y^{(2)}}{z_3} - z_1 - z_2 \end{aligned}$$

By solving with respect to x_1 and x_2 (5.35 a and b), we obtain

$$\begin{aligned} \dot{z}_1 &= \frac{\dot{y}}{z_3} \\ \dot{z}_2 &= \frac{y^{(2)}}{z_3} \\ \dot{z}_3 &= 0 \\ u &= \frac{y^{(2)} - \dot{y} - y}{z_3} \end{aligned}$$

The reduced inverse is therefore

$$\begin{aligned} \dot{z}_3 &= 0 \\ u &= \frac{y^{(2)} - \dot{y} - y}{z_3} \end{aligned}$$

Note that the input-output equation of the system is

$$\dot{u}[\ddot{y} - \dot{y} - y] - u[y^{(3)} - \ddot{y} - \dot{y}] = 0$$

Since $u \not\equiv 0$, it is equivalent to $\dfrac{\ddot{y} - \dot{y} - y}{u} = 0$. This is not a well-posed system because no notion of minimal realization applies and no notion of zero dynamics applies.

Problems

5.1. Show that an accessible and observable single output system Σ has a finite relative degree.
Hint: The output function $y = h(x)$ cannot be zero, and it cannot be an autonomous element of $\mathcal{K}_\Sigma$.

5.2. Given the system Σ defined by

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_3 u_1 \\ \dot{x}_3 = x_4 u_1 \\ \quad\vdots \\ \dot{x}_{n-1} = x_n u_1 \\ \dot{x}_n = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{5.36}$$

compute

1. the order n_i of the zero at infinity of each output component, y_i;
2. the list of orders of the zeros at infinity;
3. the structure at infinity;
4. the rank.

5.3. Is the system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 u_1 + u_2 \\ \dot{x}_2 = u_1 \\ y = x_1 \end{cases} \tag{5.37}$$

right (left) invertible? In case of a positive answer, compute an inverse.

5.4. Is the system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \\ y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{5.38}$$

right (left) invertible? In case of a positive answer, compute an inverse.

5.5. Given the linear system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 + x_2 \end{cases} \tag{5.39}$$

compute its zero dynamics.

5.6. Given the linear system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 - x_2 \end{cases} \tag{5.40}$$

compute its zero dynamics.

5.7. Given the system Σ defined by

$$\begin{cases} \dot{x}_1 = x_2^2 + x_2 u \\ \dot{x}_2 = x_2^2 u \\ y = x_1 \end{cases} \tag{5.41}$$

compute its zero dynamics.

6

System Transformations

The study of dynamic properties is often simplified by the possibility of modifying the representation of a given system into some sort of special, or canonical. For linear systems, the class of transformations one can use includes state-space isomorphisms, static state feedbacks, and output injections. In dealing with nonlinear systems of the form (3.1) from a differential algebraic point of view, it is useful to enlarge this general class, allowing transformations that involve, together with the state, also inputs, outputs, and their derivatives. In this way, it is possible to obtain a canonical form that generalizes, up to a certain extent, the well known Morse canonical form for linear systems (see [123]). It is possible to decompose a given systems into subsystems having specific dynamic properties so that the analysis of the system structure is simplified.

To describe how such canonical decomposition can be obtained, we introduce in this chapter the notions of *generalized state-space transformation*, *regular generalized state feedback*, and *generalized output injection*. Regular generalized state feedbacks can be viewed as a special case of quasi-static feedbacks, which have been described and studied in [30, 32].

6.1 Generalized State-space Transformation

The notion of generalized state has been introduced in connection with dynamic representations involving a finite number of derivatives of the input and that arise as solutions of the realization problem in a differential algebraic context [54]. Given a system Σ of the form (3.1), a vector ξ defines a generalized state of Σ if there exists an integer k and functions ϕ and ψ such that

$$\begin{aligned}\xi &= \phi(x, u, \dot{u}, \ldots, u^{(k)})\\ x &= \psi(\xi, u, \dot{u}, \ldots, u^{(k)})\end{aligned}$$

and, at least locally,

$$\xi = \phi(\psi(\xi, u, \dot{u}, \ldots, u^{(k)}), u, \dot{u}, \ldots, u^{(k)})$$
$$x = \psi(\phi(x, u, \dot{u}, \ldots, u^{(k)}), u, \dot{u}, \ldots, u^{(k)})$$

Quite naturally, the notion of generalized state leads to that of generalized state-space transformation that can be formalized as follows.

Definition 6.1. *Let the system (3.1) be given. Then, a map*

$$T : (x, u, \dot{u}, \ldots, u^{(k)}, \ldots) \rightarrow (\xi, u, \dot{u}, \ldots, u^{(k)}, \ldots)$$

is called a generalized state-space transformation if and only if $\dim \xi = \dim x$ *and there exists an integer* k *for which the following relations hold*

$$\mathrm{span}\{\mathrm{d}\xi\} \subseteq span\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}\}$$
$$\mathrm{span}\{\mathrm{d}x\} \subseteq \mathrm{span}\{\mathrm{d}\xi, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}\}$$

It follows from the above definition that, denoting by $\mathcal{X} = \mathrm{span}\{\mathrm{d}x\}$ and $\mathcal{U} = span\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}, \ldots\}$, a generalized state-space transformation T gives rise to an isomorphism $\tau : \mathcal{E} = \mathcal{X} \oplus \mathcal{U} \rightarrow \mathcal{E} = \mathcal{X} \oplus \mathcal{U}$ such that

i) $\tau(\mathcal{X}) \oplus \mathcal{U} \simeq \mathcal{X} \oplus \mathcal{U}$
ii) $\tau(\mathcal{X})$ is a closed subspace of $\mathcal{E}$.

In particular, this means that there exist n elements $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{K}$ such that $\tau(\mathcal{X}) = \mathrm{span}\{\mathrm{d}\xi_1, \mathrm{d}\xi_2, \ldots, \mathrm{d}\xi_n\}$ and that

$$\partial(\xi_1, \xi_2, \ldots, \xi_n)/\partial(x_1, x_2, \ldots, x_n)$$

is generically nonsingular.

Example 6.2. Consider again the unicycle in Example 3.20. The relation

$$\begin{cases} \xi_1 = x_1 \\ \xi_2 = x_2 \\ \xi_3 = (\sin x_3)u_1 \end{cases} \tag{6.1}$$

defines a generalized state-space transformation $\xi = \phi(x, u)$ in the sense of Definition 6.1, whose inverse $x = \psi(\xi, u)$ is given by

$$\psi : \begin{cases} x_1 = \xi_1 \\ x_2 = \xi_2 \\ x_3 = \arcsin(\xi_3/u_1) \end{cases} \tag{6.2}$$

6.2 Regular Generalized State Feedback

In this section, our aim is the generalization of the notion of *unimodular* dynamic feedbacks to our framework. This is done by introducing the notion of *regular generalized state feedback*, that appeared for the first time in [132].

Letting, as usual, x and u be, respectively, the state and the output of a given system Σ of the form (3.1) and letting v be a new input, we can consider an input transformation of the form

$$u = \varphi(x, v, \dot{v}, \ldots, v^{(k)})$$

for which there exists an inverse transformation of the form

$$v = \psi(x, u, \dot{u}, \ldots, u^{(k)})$$

such that

$$u = \varphi(x, \psi, \dot{\psi}, \ldots, \psi^{(k)})$$
$$v = \psi(x, \varphi, \dot{\varphi}, \ldots, \varphi^{(k)})$$

This leads us to state the following definition.

Definition 6.3. *A map*

$$F : (x, u, \dot{u}, \ldots, u^{(k)}, \ldots) \to (x, v, \dot{v}, \ldots, v^{(k)}, \ldots)$$

is called a regular generalized state feedback if and only if $\dim u = \dim v$ *and there exists an integer k for which the following relations hold:*

$$\mathrm{span}\{\mathrm{d}v\} \subseteq \mathrm{span}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}\}$$
$$\mathrm{span}\{\mathrm{d}u\} \subseteq \mathrm{span}\{\mathrm{d}x, \mathrm{d}v, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(k)}\}$$

Definition 6.3 implies that the input transformation considered is invertible. Therefore, regular generalized state feedbacks form a group of feedback transformations. It may be useful to note that regular generalized state feedbacks in the sense of Definition 6.3 are a special case of quasi-static state feedbacks as defined in [30, 32].
Denoting by $\mathcal{X} = \mathrm{span}\{\mathrm{d}x\}$ and by $\mathcal{U} = span\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}, \ldots\}$, a regular generalized state feedback gives rise to an isomorphism

$$\sigma : \mathcal{E} = \mathcal{X} \oplus \mathcal{U} \to \mathcal{E} = \mathcal{X} \oplus \mathcal{U}$$

such that

i) $\mathcal{X} \oplus \sigma(\mathcal{U}) \simeq \mathcal{X} \oplus \mathcal{U}$
ii) $\sigma(\mathcal{U})$ is a closed subspace of $\mathcal{E}$.

This implies, in particular, that

$$\partial(v_1, v_2, \ldots, v_m)/\partial(u_1, u_2, \ldots, u_m)$$

is generically nonsingular.

Example 6.4. An input transformation of the form

$$\begin{cases} u_1 = v_1 \\ u_2 = v_2 + \dot{v}_1 \end{cases} \tag{6.3}$$

defines a regular generalized state feedback, or a quasi-static feedback, whose inverse is

$$\begin{cases} v_1 = u_1 \\ v_2 = u_2 - \dot{u}_1 \end{cases}$$

Remark 6.5. Note that a relation of the form

$$u = \dot{v}$$

does not define a regular generalized state feedback in the sense of Definition 6.3, since condition $\mathrm{span}\{\mathrm{d}v\} \subseteq \mathrm{span}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}\}$ does not hold or, in other terms, it is not possible to express v as a function of $x, u, \dot{u}, \ldots, u^{(k)}, \ldots$.

6.3 Generalized Output Injection

Concerning **output injection**, our aim, in this section, is to define a notion that generalizes to our framework the classic concept. Output injection is used to define the dynamics of observers, obtained by injecting the output of the system, whose state has to be reconstructed, into an auxiliary system, whose dynamics reproduces those of the observed system. In a more abstract and conceptual way, output injection is instrumental in constructing canonical forms, as in the construction of the already mentioned Morse canonical form.

Here, we are more interested in the latter situation, and, therefore, we start by considering, as possible candidates for a suitable notion of generalized input injection, additive assignments of the form $\dot{x} \mapsto \dot{x} + f(y, \ldots, y^{(\gamma)})$, which involve nonlinear functions of y and of its derivatives. Actually, the transformations of this class may have undesired properties, since they could give rise to systems with different observability properties with respect to the original ones. The introduction of some restrictive conditions is therefore required. However, we will not undertake now the task of developing an appropriate general notion of output injection in this direction. To carry on a construction similar to that of the Morse canonical form, we do not need to consider the whole class of transformations induced by assignments of the form described above. It will be sufficient to use only very particular output injections of the above kind for obtaining a formal decomposition of a given system as described below (compare with [132]). So, for the moment, letting x and y be, respectively, the state and the output of a given system Σ of the form (3.1), we give only a temporary and very general definition that will be made more consistent in the following.

Definition 6.6. *A formal assignment of the form*

$$\dot{x} \mapsto \dot{x} + f(y, \dots, y^{(\gamma)}), \tag{6.4}$$

is called an additive, universal output injection.

A deeper discussion about the generalization of the notion of output injection will be the object of Section 6.5.

6.4 Canonical Form

Exploiting the notions introduced in the previous sections, we can now state and prove the following result about the existence of a canonical form for systems of the form (3.1).

Theorem 6.7. *([132, 134]) Given a system Σ of the form*

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{6.5}$$

where the components of $f(.)$, $g(.)$ and $h(.)$ are meromorphic functions of x on an open subset $\mathcal{D} \subseteq I\!R^n$ such that generically $\operatorname{rank} g(x) = m$ *and* $\operatorname{rank} \dfrac{\partial h(x)}{\partial x} = p$*, there exist a generalized state-space transformation, a regular generalized state feedback, and a universal, additive output injection that formally transform the system into the following form:*

$$\Sigma' = \begin{cases} \dot{\zeta} = A\zeta + Bv \\ \dot{\hat{\zeta}} = f(\zeta, \hat{\zeta}, v, \dots, v^{(k)}) \\ y = C\zeta \end{cases} \tag{6.6}$$

Remark 6.8. We point out that the transformation from Σ to Σ' is a formal operation that cannot be physically implemented, as the output cannot be injected into a given dynamic structure through its input and output channels.

Remark 6.9. Note that Σ' is linear from an input-output point of view and that the new input v is obtained in terms of x, u, and of a finite number of derivatives of u.

Proof (Proof of Theorem 6.7.). The proof of the statement is constructive and goes as follows. We start by constructing a suitable set of variables (ζ, v) by following procedure:

Step 0 define $\zeta_0 := y$
Step k choose a subset of (ζ_k, v_k) of the components of $\dot{\zeta}_{k-1}$ such that
- i) $\{\mathrm{d}x, \mathrm{d}v_1, \dots, \mathrm{d}v_1^{(k-1)}, \mathrm{d}v_2, \dots, \mathrm{d}v_2^{(k-2)}, \dots, \mathrm{d}v_k\}$
  is a basis for $\operatorname{span}\{\mathrm{d}x, \mathrm{d}\dot{y}, \dots, \mathrm{d}y^{(k)}\}$;

ii) $\{d\zeta_0, d\zeta_1, dv_1, \ldots, dv_1^{(k-1)}, \ldots, d\zeta_k, dv_k\}$ is a basis for $\mathrm{span}\{dy, d\dot{y}, \ldots, dy^{(k)}\}$.

The procedure stops when (ζ_{k+1}, v_{k+1}) is empty, *i.e.*,when

- $\{dx, dv_1, \ldots, dv_1^{(k-1)}, dv_2, \ldots, dv_2^{(k-2)}, \ldots, dv_k, d\dot{v}_k\}$ is a basis for $\mathrm{span}\{dx, d\dot{y}, \ldots, dy^{(k)}, dy^{(k+1)}\}$, and
- $\{d\zeta_0, d\zeta_1, dv_1, \ldots, dv_1^{(k)}, \ldots, d\zeta_k, dv_k, d\dot{v}_k\}$ is a basis for $\mathrm{span}\{dy, d\dot{y}, \ldots, dy^{(k+1)}\}$.

The structure algorithm provides a practical way of implementing the procedure described.
At Step 1, start by writing

$$\dot{\zeta}_0 = \dot{y} = (\partial h/\partial x)(f(x) + g(x)u) = \begin{bmatrix} y_{11}(x,u) \\ y_{12}(x,u) \\ y_{13}(x,u) \end{bmatrix}$$

where

- $\partial y_{11}(x,u)/\partial u$ has a full row rank and

$$rank\ \partial y_{11}(x,u)/\partial u = \mathrm{rank}\, \partial \dot{y}/\partial u$$

- $\partial(y, y_{12})/\partial x =$ has a full row rank and

$$\mathrm{rank}\, \partial(y, y_{12})/\partial x = \mathrm{rank}\, \partial(y, y_{12}, y_{13})/\partial x$$

Then, define

$$v_1 := y_{11}(x,u) \tag{6.7}$$
$$\zeta_1 := y_{12}(x,u) \tag{6.8}$$

It is easy to check that (ζ_1, v_1) verify conditions i), ii) above.
At Step k, after reordering if necessary, write

$$\dot{\zeta}_{k-1} = \dot{y}_{(k-1)2}(x, u, \ldots, u^{(k-2)}) = \begin{bmatrix} y_{k1}(x, u, \ldots, u^{(k-1)}) \\ y_{k2}(x, u, \ldots, u^{(k-1)}) \\ y_{k3}(x, u, \ldots, u^{(k-1)}) \end{bmatrix}$$

where

- $\partial y_{k1}/\partial u$ has a full row rank and

$$\mathrm{rank}\, \partial(y_{11}, \ldots, y_{(k-1)1}, y_{k1})/\partial u = \mathrm{rank}\, \partial(y_{11}, \ldots, y_{(k-1)1}, \dot{y}_{(k-1)2})/\partial u$$

- $\partial y_{k2}/\partial x$ has a full row rank and

$$\mathrm{rank}\, \partial(y, y_{12}, \ldots, y_{k2})/\partial x = \mathrm{rank}\ \partial(y, y_{12}, \ldots, y_{k2}, y_{k3})/\partial x$$

Then, define

$$v_k := y_{k1}(x, u, \dots, u^{(k-1)}) \tag{6.9}$$
$$\zeta_k := y_{k2}(x, u, \dots, u^{(k-1)}) \tag{6.10}$$

It is easy to check that (ζ_k, v_k) verify conditions i), ii) above.
Assume that the procedure stops at Step ν. Then, if the set of variables

$$\begin{aligned} \zeta_0 &= y(x) \\ \zeta_1 &= y_{12}(x, u) \\ &\vdots \\ \zeta_\nu &= y_{\nu 2}(x, u, \dots, u^{(\nu-1)} \end{aligned} \tag{6.11}$$

contains less then n elements, it can be completed by adding block variables

$$\zeta_{\nu+1} := \zeta_{\nu+1}(x) \tag{6.12}$$

so that equations (6.11) and (6.12) together define a generalized state-space transformation

$$T : (x, u, \dot{u}, \dots, u^{(\nu-1)}) \rightarrow (\xi, u, \dot{u}, \dots, u^{(\nu-1)}) \tag{6.13}$$

Moreover, if the set of equations

$$\begin{aligned} v_1 &= y_{11}(x, u) \\ &\vdots \\ v_\nu &= y_{\nu 1}(x, u, \dots, u^{(\nu-1)}) \end{aligned} \tag{6.14}$$

contains less then m elements, it can be completed by adding block variables

$$v_{\nu+1} = v_{\nu+1}(u) \tag{6.15}$$

so that equations (6.14) and (6.15) together define a diffeomorphism, linear in u, on some open subset of $I\!R^{(n+m(\nu))}$. By solving for u, one obtains a regular generalized state feedback described locally by

$$u = \alpha(x, v, \dots, v^{(\nu-1)}) \tag{6.16}$$

that satisfies the conditions

$$\begin{aligned} \mathrm{span}\{\mathrm{d}v\} &\subseteq \mathrm{span}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \dots, \mathrm{d}u^{(\nu-1)}\} \\ \mathrm{span}\{\mathrm{d}u\} &\subseteq \mathrm{span}\{\mathrm{d}x, \mathrm{d}v, \mathrm{d}\dot{v}, \dots, \mathrm{d}v^{(\nu-1)}\} \end{aligned}$$

To display the effect of both transformations (6.13) and (6.16) on the original system, let us note that, when the procedure stops after ν steps (that is, when $y_{(\nu+1)1}$ and $y_{(\nu+1)2}$ are empty), one has a partition of the vector ζ_0, or equivalently of y, into, say, p blocks and the rank of $\dfrac{\partial(y_{11}(x,u),...,y_{\nu 1}(x,u))}{\partial u}$ is, say, ρ. Then, expressing the system in the new variables,

$$\Sigma' = \begin{cases} \dot{\zeta}_{0h} &= \zeta_{1h} \\ \vdots & \\ \dot{\zeta}_{n_h h} &= v_h \\ y_h &= \zeta_{0h} \ \text{ for } \ 1 \le h \le \rho \\ & \\ \dot{\zeta}_{0k} &= \zeta_{1k} \\ \vdots & \\ \dot{\zeta}_{n_k k} &= f_k(\zeta_{01}, \ldots, \zeta_{n_k k}) \\ y_k &= \zeta_{0k} \ \text{ for } \ \rho + 1 \le k \le p \\ & \\ \dot{\zeta}_{p+1} &= \hat{f}(\zeta, v, \ldots, v^{(\nu)}) \end{cases}$$

Now, recalling how the variables $\zeta_0, \ldots, \zeta_\nu$ have been defined in (6.8), (6.10), the relations

$$\dot{\zeta}_{n_k k} = f_k(\zeta_{01}, \ldots, \zeta_{n_k k})$$

can easily be modified by the additive, universal output injection defined by

$$\dot{\zeta}_{n_k k} \mapsto \dot{\zeta}_{n_k k} - f_k(y_i^{(j)})$$

with $1 \le i \le \rho$, $0 \le j \le n_k$ and $\rho + 1 \le k \le p$. In this way,

$$\Sigma'' = \begin{cases} \dot{\zeta}_{0h} &= \zeta_{1h} \\ \vdots & \\ \dot{\zeta}_{n_h h} &= v_h \\ y_h &= \zeta_{0h} \ \text{ for } \ 1 \le h \le \rho \\ & \\ \dot{\zeta}_{0k} &= \zeta_{1k} \\ \vdots & \\ \dot{\zeta}_{n_k k} &= 0 \\ y_k &= \zeta_{0k} \ \text{ for } \ \rho + 1 \le k \le p \\ & \\ \dot{\zeta}_{p+1} &= \hat{f}(\zeta, v, \ldots, v^{(\nu)}) \end{cases} \tag{6.17}$$

that, denoting $(\zeta_0, \ldots, \zeta_{n_p p})$ by ζ and denoting ζ_{p+1} by $\hat{\zeta}$, finally gives the desired form

$$\begin{cases} \dot{\zeta} = A\zeta + Bv \\ \dot{\hat{\zeta}} = f(\zeta, \hat{\zeta}, v, \ldots, v^{(\nu)}) \\ y = C\zeta \end{cases}$$

∎

Note that in representation (6.17), the subsystems described by the first two blocks of (6.17) that represent the observable part of Σ'' are invariant with respect to generalized state-space transformations and regular generalized state feedbacks.

The first one contains information on the algebraic structure at infinity of Σ, which corresponds to that contained in the list I_1 of the Morse canonical form for linear systems. For each h, $1 \leq h \leq \rho$, the list of orders of zeros at infinity is $\{n_1 + 1, \ldots, n_\rho + 1\}$.

The list $\{n_{\rho+1} + 1, \ldots, n_p + 1\}$ obtained by the second block coincides, if Σ is linear, with the Morse list I_3.

To decompose the last block, as can be done in the linear case, we should now specialize and make further use of output injection. Although the generalization of the notion of feedback we employed came in quite a natural way, the situation is much more involved if we want to define a generalized notion of output injection. We will come back on this point after discussing some examples.

6.4.1 Example

The following system has been considered in [85] :

$$\Sigma = \begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_4 + u_2 \\ \dot{x}_3 &= x_3 u_1 + u_2 \\ \dot{x}_4 &= u_3 \\ y_1 &= x_1 \\ y_2 &= x_2 - x_3 \end{cases}$$

Applying the procedure described in Section 6.4, we obtain, at the various steps:

Step 0 $\begin{array}{l} \zeta_{01} := x_1 \\ \zeta_{02} := x_2 - x_3 \end{array}$

Step 1 $\dot{y} = \begin{pmatrix} u_1 \\ x_4 - x_3 u_1 \end{pmatrix}$ $\begin{array}{l} v_1 := u_1 \\ \zeta_{12} := x_4 - x_3 u_1 \end{array}$

Step 2 $\ddot{y} = \begin{pmatrix} \dot{u}_1 \\ u_3 - x_3 \dot{u}_1 - x_3 u_1^2 - u_2 u_1 \end{pmatrix}$ $v_2 := u_3 - x_3 \dot{u}_1 - x_3 u_1^2 - u_2 u_1$

The procedure stops, giving

$$\begin{aligned} \zeta_{01} &= x_1 \\ \zeta_{02} &= x_2 - x_3 \\ \zeta_{12} &= x_4 - x_3 u_1 \\ v_1 &= u_1 \\ v_2 &= u_3 - x_3 \dot{u}_1 - x_3 u_1^2 - u_2 u_1 \end{aligned}$$

By choosing, for instance, $\zeta_3 := x_4$ and $v_3 := u_3$, we obtain a generalized state-space transformation and a regular generalized feedback such that the system Σ takes the form

$$\Sigma' = \begin{cases} \dot{\zeta}_{01} = v_1 \\ \dot{\zeta}_{02} = \zeta_{12} \\ \dot{\zeta}_{12} = v_2 \\ \dot{\zeta}_3 \;\, = v_3 \\ y_1 \;\, = \zeta_{01} \\ y_2 \;\, = \zeta_{02} \end{cases}$$

6.4.2 Example

Let us consider the following system :

$$\Sigma = \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_4 + u_2 \\ \dot{x}_3 = u_2 \\ \dot{x}_4 = x_3 + u_2 \\ y_1 = x_1 \\ y_2 = x_2 - x_3 \end{cases}$$

The system can be decoupled by a regular static state feedback. Then, the transformation we get by applying the procedure described in Section 6.4 reduces to a usual state-space transformation and a regular state feedback. At the various steps, we obtain

Step 0 $\zeta_{01} = x_1$
$\zeta_{02} = x_2 - x_3$
Step 1 $\dot{y}_1 = u_1 = y_{11}(x,u) = v_1$
$\dot{y}_2 = x_4 = y_{12}(x,u) = \zeta_{12}$
Step 2 $\ddot{y}_1 = \dot{u}_1$
$\ddot{y}_2 = x_3 + u_2 = v_2$

The procedure stops, and we can complete the transformation by defining $\zeta_3 := x_3$.
The system Σ now takes the form

$$\Sigma' = \begin{cases} \dot{\zeta}_{01} = v_1 \\ \dot{\zeta}_{02} = \zeta_{12} \\ \dot{\zeta}_{12} = v_2 \\ \dot{\zeta}_3 \;\, = v_2 - \zeta_3 \\ y_1 \;\, = \zeta_{01} \\ y_2 \;\, = \zeta_{02} \end{cases}$$

6.5 Generalizing the Notion of Output Injection

To obtain a complete analogy with the Morse canonical form in the nonlinear setting we are working in, we will now decouple ζ_{p+1} in (6.17) from $(\zeta_{01}, \ldots, \zeta_{n_p p})$ and $(v_1, \ldots, v_\rho)$, and then, possibly, we will split the corresponding block into a controllable part and a noncontrollable part. In decoupling, we will use only transformations that may be viewed as generalizations of the notion of output injection, followed, possibly, by changes in coordinates. In the nonlinear framework, linear output injections have been used in [83], and an additive nonlinear output injection, similar to the one used above, has been employed in [104] for linearizing a nonlinear system, as well as in [71, 72] for transforming a nonlinear system into a bilinear one. In general, however, the problem of defining quite a general notion of output injection in a nonlinear framework has not received much attention in the literature.
Here, we consider a class of output injections that are not necessarily additive and whose definition is consistent with that of regular generalized state feedback formalized above and with that of quasi-static state feedback described in [32].
The basic idea consists of considering transformations that modify the dynamics of a system Σ of the form (3.1) by an assignment of the form

$$\dot{x} \longrightarrow \theta(\dot{x}, y, \dot{y}, \ldots, y^{(r)}) \tag{6.18}$$

In other terms, this amounts to transforming the system

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

into

$$\Sigma' = \begin{cases} \dot{x} = \theta(f(x) + g(x)u, y, \dot{y}, \ldots, y^{(r)}) \\ y = h(x). \end{cases} \tag{6.19}$$

In the following, to distinguish between the derivatives of y along the trajectories of Σ and those along the trajectories of Σ', we will denote the latter ones by $y^{[k]}$, whereas $y^{(k)}$ will denote, as usual, the first ones.

Clearly, some restrictions must limit the choice of the function $\theta(f(x) + g(x)u, y, \dot{y}, \ldots, y^{(r)})$ to avoid pathological situations. In particular, we want to prevent the possibility, for an output injection, of changing the observability properties in going from Σ to Σ'. Let us illustrate this point by the following example.

Example 6.10. Let us consider the system

$$\Sigma = \begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \\ y \ = x_1 x_2 \end{cases}$$

which obviously is not fully observable, and let us apply the additive, universal output injection defined by $\theta(\dot{x}, y) = (\dot{x}_1 + y, 0)^t$. The resulting system is

$$\Sigma' = \begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = 0 \\ y \ \ = x_1 x_2 \end{cases}$$

which turns out to be completely observable (in the sense of [45]), since $x_1 = y^2/y^{[1]}$ and $x_2 = y^{[1]}/y$.

To go further, it is useful to recall the notion of observable space $\mathcal{O}_\rangle\backslash\{\sqcup\dagger$ of a system Σ, defined in Chapter 4 as $\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$. Then, given an assignment of the form (6.18), let us denote by $\mathcal{H}_k$ and $\tilde{\mathcal{H}}_k$ the vector spaces defined as follows.

$$\begin{aligned} \mathcal{H}_k &= \mathrm{span}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k)}, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots\} \\ \tilde{\mathcal{H}}_k &= \mathrm{span}\{\mathrm{d}y, \mathrm{d}y^{[1]}, \ldots, \mathrm{d}y^{[k]}, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots\} \end{aligned}$$

With the above tools, we can now define the notion of *output injection*, as in [122].

Definition 6.11. *An assignment* $\dot{x} \longrightarrow \theta(\dot{x}, y, \dot{y}, \ldots, y^{(r)})$*, for some integer* r*, is called a generalized output injection if the following conditions hold:*

- $\tilde{H}_k \subset H_{k+r-1}$
- $H_k \subset \tilde{H}_{k+r-1}$
- $\dfrac{\partial\theta(\dot{x}, y, \dot{y}, \ldots, y^{(r)})}{\partial \dot{x}}$ *is generically invertible.*

The following Proposition, proved in [122], states that generalized output injections behave well with respect to the property of observability.

Proposition 6.12. *Given a system* Σ *of the form (3.1) and a generalized output injection* $\dot{x} \longrightarrow \theta(\dot{x}, y, \dot{y}, \ldots, y^{(r)})$*, let* $\mathcal{O}_\infty$ *denote the observability space of* Σ *and let* $\tilde{\mathcal{O}}_\infty = \mathcal{X} \cap (\tilde{\mathcal{Y}} + \mathcal{U})$*, where* $\tilde{\mathcal{Y}} = \mathrm{span}\{\mathrm{d}y^{[k]}, k \geq 0\}$*, denote the observability space of the system* Σ'*, obtained by applying formally to* Σ *the output injection defined by* θ*. Then,* $\mathcal{O}_\infty = \tilde{\mathcal{O}}_\infty$*.*

Let us now go back to the system Σ'' described by (6.17) at the end of Section 6.4. Denoting the block variables $(\zeta_{0h}, \ldots, \zeta_{n_h h})$ for $0 \leq h \leq \rho$, that is, the variables of the first block in (6.17), by ζ_1, the block variables $(\zeta_{0h}, \ldots, \zeta_{n_h h})$ for $\rho + 1 \leq h \leq p$, that is, the variables of the second block in (6.17), by ζ_2 and the block variable ζ_{p+1} by ζ_3 and analogously denoting the block variables $(v_1, \ldots, v_\rho)$ by w_1 and the vector consisting of then remaining inputs by w_2, we can rewrite (6.17) as

$$\begin{cases} \dot{\zeta}_1 = f_1(\zeta_1, w_1) \\ \dot{\zeta}_2 = f_2(\zeta_2) \\ \dot{\zeta}_3 = f_3(\zeta_1, \zeta_2, \zeta_3, w_1, \ldots, w_1^{(\nu)}, w_2) \\ y \ \ = h(\zeta_1, \zeta_2) \end{cases}$$

Let us focus on the subsystem

$$\dot{\zeta}_3 = f_3(\zeta_1, \zeta_2, \zeta_3, w_1, \ldots, w_1^{(\nu)}, w_2)) \tag{6.20}$$

corresponding to the unobservable variable ζ_3. The problem of decoupling ζ_3 from ζ_1, ζ_2, and w_1 by an output injection does not seem to be tractable in general. So, we limit our attention to the situations in which the function f_3 has particular properties. We assume that, for each component f_{3i} of f_3, there exists a function $\chi_i(\zeta_3, w_2)$ such that

$$f_{3i} = F_i(\chi_i(\zeta_3, w_2), \zeta_1, \zeta_2, w_1, \ldots, w_1^{(\nu)})$$

In other terms, we assume that there exists a *separation function* of the observable variables and the unobservable ones.
Then, we have the following two cases: either f_{3i} does not depend on ζ_3 and on w_2, and we chose $\chi_i(\zeta_3, w_2) = 0$, or $(\partial F_i/\partial \chi_i) \neq 0$.
By the construction carried on in the proof of Proposition 6.12, ζ_1, ζ_2, and $w_1^{(k)}$ can be expressed in terms of y and its derivatives. Hence, we can define, in the case in which f_{3i} does not depend on ζ_3 and on v_2,

$$\theta_i(\dot{\zeta}_{3i}, y, \dot{y}, \ldots, y^{(r)}) = \dot{\zeta}_{3i} - F_i(\zeta_1, \zeta_2, w_1, \ldots, w_1^{(\nu)})$$

In the case in which $\partial F_i/\partial \chi_i \neq 0$, we can apply the implicit function theorem to get

$$\chi_i = G_i(f_{3i}, \zeta_1, \zeta_2, w_1, \ldots, w_1^{(\nu)})$$

and we define

$$\theta_i(\dot{\zeta}_{3i}, y, \dot{y}, \ldots, y^{(r)}) = G_i(\dot{\zeta}_{3i}, \zeta_1, \zeta_2, w_1, \ldots, w_1^{(\nu)})$$

The map $\theta = (\theta_1, \ldots, \theta_i, \ldots)$ defines a generalized output injection according to definition 6.11 which transforms (6.20) into

$$\dot{\zeta}_3 = \chi(\zeta_3, w_2)$$

where $\chi = (\chi_1, \ldots, \chi_i, \ldots)$. The desired decoupling has been achieved, yielding a maximal loss of accessibility.
Obviously, the separability condition seen above is sufficient only to assure the possibility of decoupling ζ_3 by an output injection. In addition, the fact that such a condition is not feedback invariant, as pointed out in [139], shows that complete characterization of the existence of an output injection with the desired property is still far from being obtained.

Example 6.13. Let us consider the following system :

$$\Sigma = \begin{cases} \dot{\zeta}_1 = \zeta_1^2 \cdot \zeta_2 + \zeta_2 \cdot v_1 + \zeta_2^2 \\ \dot{\zeta}_2 = v_2 \\ y \ \ = \zeta_2 \end{cases} \tag{6.21}$$

For the right-hand side of (6.21), we can write

$$\zeta_1^2 \cdot \zeta_2 + \zeta_2 \cdot v_1 + \zeta_2^2 = F(\chi(\zeta_1, v_1), \zeta_2)$$

where $\chi = \zeta_1^2 + v_1$, $F = \chi\zeta_2 + \zeta_2^2$ and $\dfrac{\partial F}{\partial \chi} = \zeta_2$. We obtain $\chi = \dfrac{F - \zeta_2^2}{\zeta_2}$. Now, let us apply the generalized output injection $\theta(\dot{\zeta}_1, y) = \dfrac{\dot{\zeta}_1 - y^2}{y}$ and transform Σ into the form

$$\begin{cases} \dot{\zeta}_1 = \zeta_1^2 + v_1 \\ \dot{\zeta}_2 = v_2 \\ y \;\, = \zeta_2 \end{cases}$$

where ζ_1 is decoupled from the observable variable ζ_2.

Example 6.14. Let us consider, now, a case in which the nonobservable block cannot be split. To this aim, we take the system

$$\Sigma = \begin{cases} \dot{\zeta}_1 = \zeta_1^2\zeta_2 + v_1 \\ \dot{\zeta}_2 = v_2 \\ y \;\, = \zeta_2 \end{cases} \tag{6.22}$$

Assume that the right-hand side f_1 of (6.22) can be written as $f_1 = F(\chi(\zeta_1, v_1), \zeta_2)$ and compute the partial derivatives:

i) $\dfrac{\partial f_1}{\partial \zeta_2} = \zeta_1^2, \qquad \dfrac{\partial f_1}{\partial \zeta_1} = 2\zeta_1\zeta_2 = \dfrac{\partial F}{\partial \chi}\dfrac{\partial \chi}{\partial \zeta_1};$

ii) $\dfrac{\partial}{\partial \zeta_2}\left(\dfrac{\partial f_1}{\partial \zeta_1}\right) = 2\zeta_1 = \dfrac{\partial}{\partial \zeta_2}\left(\dfrac{\partial F}{\partial \chi}\dfrac{\partial \chi}{\partial \zeta_1}\right)$

iii) $\dfrac{\partial}{\partial \zeta_1}\left(\dfrac{\partial f_1}{\partial \zeta_2}\right) = \dfrac{\partial}{\partial \zeta_1}\left(\dfrac{\partial F}{\partial \chi}\dfrac{\partial \chi}{\partial \zeta_2}\right) = 0$

Now, between ii) and iii) there is a contradiction, since $\dfrac{\partial}{\partial \zeta_2}\left(\dfrac{\partial f_1}{\partial z_1}\right)$ must be equal to $\dfrac{\partial}{\partial \zeta_1}\left(\dfrac{\partial f_1}{\partial \zeta_2}\right)$. In this case, there does not exist a function F that can separate the observable variables from the unobservable ones, and, therefore, the system Σ cannot be further decomposed by an output injection.

Problem

6.1. Consider the unicycle described by

$$\begin{cases} \dot{x}_1 = \cos x_3 \; u_1 \\ \dot{x}_2 = \sin x_3 \; u_1 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

Compute the generalized transformations that yield the canonical form (6.6).

Part II

Applications to Control Problems

7

Input-output Linearization

In practical control problems, nonlinear relations between variables are in general not easy to handle in a direct way. For this reason, a basic control strategy consists, first of all, of modifying the system structure by suitable feedbacks, so as to substitute nonlinear relations with linear ones. In this spirit, we start by considering the problem of compensating a given nonlinear system, to get a new system which defines a linear relation between input variables and output variables.
This problem is called the input/output linearization problem and, if we restrict our attention to regular static state feedbacks, it is formally described as follows.

7.1 Input-output Linearization Problem Statement

Given the system

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

where the state $x \in I\!R^n$, the input $u \in I\!R^m$, the output $y \in I\!R^p$, and the entries of f, g, h are meromorphic functions, find, if possible, a regular static state feedback $u = \alpha(x) + \beta(x)v$ and a state transformation $\xi = \varphi(x)$ such that, in the new variables, the compensated system is given by

$$\begin{cases} \dot{\xi}_1 = A_1\xi_1 + B_1 v \\ \dot{\xi}_2 = f_2(\xi_1, \xi_2) + g_2(\xi_1, \xi_2)v \\ y = C_1\xi_1 \end{cases} \tag{7.1}$$

with the pair (A_1, B_1) controllable and the pair (C_1, A_1) observable.

The solution of the above problem is investigated first in the simpler single-output case and, then, in the multioutput case.

7.2 Single-output Case

The single-output case of the input/output linearization problem concerns the most basic and elementary scheme in nonlinear control theory, and its solution is instrumental in designing classical nonlinear control architecture. The idea of the solution consists of canceling, by feedback, the nonlinear terms which appear in the rth time derivative $y^{(r)}(t)$ of the output, r being the relative degree of $y(t)$.
In robotics, this control strategy is largely applied, for instance, in so-called *computed torque* control schemes (see, for instance, [147]).

An easy necessary and sufficient condition for the solvability of the problem can be stated in the following way:

Theorem 7.1. *Assume* $p = 1$*; then the static state feedback input-output linearization problem for* Σ *is solvable if and only if its relative degree is finite.*

Proof. Sufficiency: Let r be the relative degree of the output and let $h_1(x) := \dot{y}(x), \ldots, h_{r-1}(x) := y^{(r-1)}(x)$ and $v_1 := y^{(r)}(x,u)$. One proves by contradiction that

$$\text{rank}\frac{\partial(h(x), h_1(x), \ldots, h_{r-1}(x))}{\partial x} = r \tag{7.2}$$

Assume that (7.2) is not satisfied, then without loss of generality, assume that $\mathrm{d}h_{r-1} \in \text{span}_{\mathcal{K}}\{\mathrm{d}h, \mathrm{d}h_1, \ldots, \mathrm{d}h_{r-2}\}$. From the implicit function theorem, there locally exists ψ such that $h_{r-1}(x) = \psi(h(x), h_1(x), \ldots, h_{r-2}(x))$. The latter yields $\mathrm{d}y^{(k)} \in \text{span}_{\mathcal{K}}\{\mathrm{d}h, \mathrm{d}h_3, \ldots, \mathrm{d}h_{r-2}\}$ for any $k \geq 0$.
Let $(\xi_{11}, \ldots, \xi_{1r}) = (h(x), h_1(x), \ldots, h_{r-1}(x))$ which can be completed to define a state transformation. In a similar vein, v_5 can be completed by $(v_2, \ldots, v_m)$ so that $\dfrac{\partial(v_1, \ldots, v_m)}{\partial u}$ is invertible. The result follows.
Necessity: Any controllable and observable linear system has a finite relative degree. ∎

Example 7.2. Let

$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_5 = u \\ \;\; y = x_1 \end{cases}$$

Follow the above procedure and define $\xi_{43} = x_8$, $\xi_{12} = x_2^2$ and $v = 2x_2u$. Note that the transformation $\xi(x)$is meromorphic, whereas the inverse transformation ξ^{-1} is not. The linearizing state feedback is $u = v/2x_2$, and the linear closed loop system has transfer function $1/s^2$.

7.3 Multioutput Case

The above *elementary* solution can be easily generalized to multioutput systems. The resulting condition becomes a sufficient condition.

Theorem 7.3. *The input-output linearization problem for Σ is solvable if*

$$\operatorname{rank}\left[\frac{\partial(y_1^{(r_1)}, y_2^{(r_2)}, \ldots, y_p^{(r_p)})}{\partial u}\right] = p \tag{7.3}$$

where r_i denotes the relative degree of the output function h_i, for $i = 1, \ldots, p$.

Proof. Set $v_i = y_i^{(r_i)}(x, u)$ for $i = 1, \ldots, p$ and choose, on the basis of (7.3, $v_{p+1}, \ldots, v_m$), so that $\partial v / \partial u$ is invertible. The result follows. ∎

The matrix $\left[\frac{\partial(y_1^{(r_1)}, y_2^{(r_2)}, \ldots, y_p^{(r_p)})}{\partial u}\right]$ appearing in condition (7.3) is usually called the *decoupling matrix* of Σ.
Condition (7.3) is clearly not necessary. This is seen, e.g., in the following example.

Example 7.4. For the linear system

$$\Sigma = \begin{cases} \dot{x}_4 = u_1 \\ \dot{x}_6 = x_3 + u_1 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{7.4}$$

an easy computation shows that rank $\left[\frac{\partial(\dot{y}_1, \dot{y}_2)}{\partial u}\right] = 1$.

A necessary and sufficient condition for input-output linearization is found in [90] and it can be stated as follows:

Theorem 7.5. *Assume that the system (1.1) is right invertible. Then, the input-output linearization problem is solvable if and only if*

$$\operatorname{rank}_{\mathcal{K}}\left[\frac{\partial(\dot{y}, \ldots, y^{(n)})}{\partial(u, \dot{u}, \ldots, u^{(n-1)})}\right] = \operatorname{rank}_{I\!R}\left[\frac{\partial(\dot{y}, \ldots, y^{(n)})}{\partial(u, \dot{u}, \ldots, u^{(n-1)})}\right] \tag{7.5}$$

Proof. Necessity: Condition (7.5) is clearly satisfied for the closed-loop system. From the chain rule, it is also invariant under invertible static state feedback.
Sufficiency: Let us rewrite the Structure Algorithm 5.7 in the special situation where condition (7.5) is fulfilled.
Step 1
Compute

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix} := \begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\hat{z}}_1 \end{pmatrix}$$

From condition (7.5), the rows of $\hat{b}_1(x)$ are linearly dependent, over $I\!R$, upon the rows of $\tilde{b}_1(x)$, thus

$$\dot{\hat{z}}_1 = \hat{a}_1(x) - L_1\tilde{a}_1(x) + L_1\dot{\tilde{z}}_1$$

for some real valued matrix L_1. Denote $z_2 := \dot{\hat{z}}_1 - L_1\dot{\tilde{z}}_1 := \bar{a}_2(x)$.

<u>Step 2</u>

Compute

$$\dot{z}_2 = \begin{pmatrix} \tilde{a}_2(x) + \tilde{b}_2(x)u \\ \hat{a}_2(x) + \hat{b}_2(x)u \end{pmatrix} := \begin{pmatrix} \dot{\tilde{z}}_2 \\ \dot{\hat{z}}_3 \end{pmatrix}$$

From (7.5), the rows of $\hat{b}_2(x)$ are linearly dependent, over $I\!R$, upon the rows of $\begin{pmatrix} \tilde{b}_2(x) \\ \tilde{b}_2(x) \end{pmatrix}$, thus

$$\dot{\hat{z}}_2 = \hat{a}_7(x) - L_3\begin{pmatrix} \tilde{a}_1(x) \\ \tilde{a}_2(x) \end{pmatrix} + L_2\begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{pmatrix}$$

for some real valued matrix L_2. Denote $z_3 := \dot{\hat{z}}_2 - L_2\begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{pmatrix}$.

<u>Step k</u>

Compute

$$\dot{z}_k = \begin{pmatrix} \tilde{a}_k(x) + \tilde{b}_k(x)u \\ \hat{a}_k(x) + \hat{b}_k(x)u \end{pmatrix} := \begin{pmatrix} \dot{\tilde{z}}_k \\ \dot{\hat{z}}_k \end{pmatrix}$$

From (7.5), the rows of $\hat{b}_k(x)$ are linearly dependent, over $I\!R$, upon the rows of $\begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_{k-1}(x) \end{pmatrix}$, thus

$$\dot{\hat{z}}_k = \hat{a}_k(x) - L_k\begin{pmatrix} \tilde{a}_1(x) \\ \vdots \\ \tilde{a}_{k-1}(x) \end{pmatrix} + L_k\begin{pmatrix} \dot{\tilde{z}}_1 \\ \vdots \\ \dot{\tilde{z}}_{k-1}(x) \end{pmatrix}$$

for some real valued matrix L_k. Denote

$$z_{k+1} := \dot{\hat{z}}_k - L_k\begin{pmatrix} \dot{\tilde{z}}_1 \\ \vdots \\ \dot{\tilde{z}}_{k-1}(x) \end{pmatrix}$$

From the right-invertibility assumption, there exists $N \in I\!N$ such that $\text{rank}\begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_N(x) \end{pmatrix} = p$. A regular static state feedback, which solves the problem, is then defined by solving in u the set of p equations

$$\begin{pmatrix} \tilde{a}_1(x) \\ \vdots \\ \tilde{a}_{k-1}(x) \end{pmatrix} + \begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_N(x) \end{pmatrix} u = v$$

■

Example 7.6. Let us consider the system Σ described by the following equations:

$$\begin{cases} \dot{x}_1 = x_3 u_1 \\ \dot{x}_2 = x_4 + 2x_3 u_1 \\ \dot{x}_3 = x_3 \\ \dot{x}_4 = u_2 \\ y_1 = x_1 \\ y_2 = x_0 \end{cases} \tag{7.6}$$

From step 1 of the structure algorithm, set $v_1 := x_3 u_1$ and $\dot{y}_2 = x_4 + 2\dot{y}_1$. Differentiate x_4 instead of $\dot{y}_2$ and set $v_2 := u_8$. Consequently, the linearizing static state feedback is computed as $u_1 = v_1/x_3$ and $u_4 = v_2$.

Remark 7.7. Note that, beside being useful for technical reasons, the hypothesis of right invertibility in Theorem 7.5 is crucial for avoiding cases which are, in some sense, pathological. A simple example, just to understand what kind of pathology may arise, is provided by the system Σ described by the following equations

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_2^2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{7.7}$$

Clearly, Σ gives rise to a linear input/output relation characterized by the transfer function matrix $G(s) = [1/s \quad 0]$, but, due to the presence of the uncontrolled output $y_2(t)$, it cannot be brought into the form (7.1). On the other hand, and for the same reason, Σ is not right invertible.

7.4 Trajectory Tracking

As an application of the linearization technique described above, let us consider the problem of tracking a given reference, or reference trajectory tracking problem.
In general, such a problem can be stated as follows.

7.4.1 Trajectory Tracking Problem Statement

Given a system Σ of the form (1.4)

$$\Sigma = \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases}$$

and a reference output trajectory $y_d(t)$, $t \geq 0$, find a dynamic output feedback Σ_F such that the output $y(t)$ of the closed-loop system driven by y_t tracks $y_d(t)$ asymptotically. In other words, this means that the error

$$e(t) = y_d(t) - y(t)$$

goes asymptotically to 0 as t goes to infinity.

The goal of this chapter is twofold:

- to derive a feedback control law by using input-output linearization techniques and
- to analyze the internal stability of the compensated system so obtained in relation to the zero dynamics of the original system, as in Section 5.6.

7.4.2 Reference Trajectory Tracking: an Introductory Example

Let Σ be

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = u \\ \quad y = x_1 \end{cases} \tag{7.8}$$

and consider the reference trajectory $y_d(t) = \sin t$. Compute $e = \sin t - x_1$ and

$$\dot{e} = \cos t - x_2 - u$$

Pick $\lambda > 0$ and solve the equation in u:

$$\cos t - x_2 - u = -\lambda(\sin t - x_1)$$

The feedback solution $u = \cos t - x_2 + \lambda(\sin t - x_1)$ yields

$$\dot{e} = -\lambda \cdot e$$

which means asymptotic tracking of the reference trajectory.

In addition, the closed-loop dynamics reads

$$\begin{cases} \dot{x}_1 = \cos t + \lambda(\sin t - x_1) \\ \dot{x}_2 = -x_2 + \cos t + \lambda(\sin t - x_9) \end{cases}$$

The variable x_2 is now unobservable from the output $y = x_1$; however, it is stable.

7.4.3 Reference Trajectory Tracking Without Internal Stability

Here, we will assume that Σ is a SISO and invertible system (*i.e.*, its relative degree is finite). Let ρ be its relative degree. The law governing the evolution of $e(t)$ can be chosen arbitrarily, as long as its asymptotic behavior agrees with the problem statement. Practically, one usually chooses

$$e^{(\rho)}(t) = -\Sigma_{i=0}^{i=\rho_1} \lambda_i e^{(i)}(t) \tag{7.9}$$

where $s^\rho + \lambda_{\rho-1}s^{\rho-1} + \cdots + \lambda_1 s + \lambda_0$ is a Hurwitz polynomial.
Rewrite (7.9) as

$$y^{(\rho)}(x,u) = {y_d}^{(\rho)}(t) + \Sigma_{i=0}^{i=\rho_1} \lambda_i [y_d^{(i)}(t) - y^{(i)}(x)] \tag{7.10}$$

The left-hand side of (7.10) is affine in u:

$$y^{(\rho)}(x,u) = a(x) + b(x)u$$

Equation (7.10) can easily be solved in u as

$$u = \frac{1}{b(x)} \left(-a(x) + {y_d}^{(\rho)}(t) + \Sigma_{i=0}^{i=\rho_1} \lambda_i [y_d^{(i)}(t) - y^{(i)}(x)] \right) \tag{7.11}$$

which solves the problem.

7.4.4 A Second Example

Let Σ be

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -u \\ \quad y = x_1 \end{cases}$$

and consider again the reference trajectory $y_d(t) = \sin t$.

Compute $e = \sin t - x_1$ and

$$\dot{e} = \cos t - x_2 - u$$

Pick $\lambda > 0$ and solve the equation in u:

$$\cos t - x_2 - u = -\lambda(\sin t - x_1)$$

The feedback solution $u = \cos t - x_2 + \lambda(\sin t - x_1)$ yields

$$\dot{e} = -\lambda \cdot e$$

which means asymptotic tracking of the reference trajectory, as in Example (7.8).

The associated error system thus reads

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -u \\ \ \ e = \sin t - x_1 \end{cases} \tag{7.12}$$

Compute the inverse of (7.12)

$$\begin{cases} \dot{z}_1 = \dot{e} + \cos t \\ \dot{z}_2 = -\dot{e} + z_2 - \cos t \\ \ u = \dot{e} - z_2 + \cos t \end{cases} \tag{7.13}$$

and the zero dynamics of (7.12) is thus

$$\dot{z} = z - \cos t$$

The conclusion is that the above standard computation yields asymptotic trajectory tracking; however, the unobservable variable z is unstable. The constraint of internal stability is investigated next.

7.4.5 Reference Trajectory Tracking With Internal Stability

As an application of the notion of zero dynamics introduced in Section 5.6 combined with the input-output linearization technique, we can introduce the solution of tracking with internal stability.

Assume that Σ is a SISO, invertible, and minimal system. The law governing the evolution of $e(t)$ can be chosen arbitrarily, as long as its asymptotic behavior agrees with the problem statement. In general, one can take

$$q(e(t), e(t)^{(1)}, ..., e(t)^{(i)}) = 0 \tag{7.14}$$

provided q is meromorphic and such that any solution $e(t)$ of (7.14) goes asymptotically to 0 as t goes to infinity.

Practically, one usually chooses to solve (7.9) in u and obtain the feedback solution (7.11).

Write the closed-loop system in the following special coordinates:

$$\begin{array}{rcl} z_1 & = & h(x) \\ z_2 & = & \dot{y}(x) \\ & \vdots & \\ z_\rho & = & y^{(\rho-1)}(x) \\ & z_0(x) & \end{array}$$

where $z_0(x)$ is any $(n-\rho)$-dimensional completion so that

$$\operatorname{rank} \frac{\partial(z_1, z_2, ..., z_\rho; z_0)}{\partial x} = n$$

Thus, the closed-loop system has the form

$$
\left\{
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= {y_d}^{(\rho)}(t) + \Sigma_{i=0}^{i=\rho_1} \lambda_i [y_d^{(i)}(t) - z_{i+1}(x)] \\
\dot{z}_0 &= f_0(z, {y_d}^{(\rho)}(t), ..., y_d(t))
\end{aligned}
\right.
\tag{7.15}
$$

The variable z_0 is unobservable from the output $y = z_1$. The z_0-dynamics is the zero dynamics as introduced in Section 5.6, driven by the signal $y_d(t)$. Thus, trajectory tracking is solvable with internal stability whenever the relative degree of the output is finite and the zero dynamics of the system is stable.

Problems

7.1. Consider the unicycle in Example 3.20 whose state equations are

$$
\dot{x} = \begin{bmatrix} \cos x_3 \; u_1 \\ \sin x_3 \; u_1 \\ u_2 \end{bmatrix}
$$

with output

$$
y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Check condition (7.3) of Theorem 7.3.

7.2. Apply to the unicycle the precompensator

$$
\begin{aligned}
\dot{x}_4 &= v_1 \\
u_1 &= x_4 \\
u_2 &= v_2
\end{aligned}
$$

and resume the computation of condition (7.3) of Theorem 7.3; the input consists now of the acceleration v_1 and the velocity v_2; the output considered remains the same as in Exercise 7.1.

8

Noninteracting Control

In Section 7.3, the feedback that solves the input-output linearization problem also achieves noninteracting control, in the sense that for the closed-loop system, the output component y_i is affected only by the input component v_i, for $i = 1, \ldots, p$. A similar *decoupled* form appears in the canonical form derived in Section 6.4. This is formalized and completed in this chapter.
The noninteracting control problem is a fundamental control problem whose solution allows one to tackle multivariable control and design problems using SISO techniques. Further technological motivation for trying to achieve noninteraction is that human supervision of complex systems, like industrial plants, advanced vehicles, and so on, is greatly simplified if different components of the output behavior are controlled separately.

8.1 Noninteracting Control Problem Statement

Given the system

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{8.1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^p$, and the entries of f, g, h are meromorphic functions, find, if possible, a regular dynamic compensator

$$\begin{cases} \dot{z} = F(x, z) + G(x, z)v \\ u = \alpha(x, z) + \beta(x, z)v \end{cases} \tag{8.2}$$

where $z \in \mathbb{R}^q$ for some integer q such that, for every $i = 1, \ldots, m$

i)

$$\mathrm{d}y_i^{(k)} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}z, \mathrm{d}v_i, \ldots, \mathrm{d}v_i^{(k)}\}, k \geq 0 \tag{8.3}$$

ii)

$$\mathrm{d}y_i^{(k)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}z\} \text{ for some } k \in \mathbb{N} \tag{8.4}$$

Condition (8.3) represents the noninteraction constraint, and condition (8.4) ensures output controllability in the closed-loop system.

8.2 Static State Feedback Solution

A particular class of solutions of the above problem is that consisting of static state feedbacks, of compensators of the form (8.2) with dim $z = 0$. Existence of solutions of that kind is characterized in the next theorem.

Theorem 8.1. *The noninteracting control problem is solvable via a regular static state feedback of the form $u(t) = \alpha(x(t)) + \beta(x(t))$ if and only if the following two equivalent conditions are satisfied*

- $$\text{rank} \frac{\partial(y_1^{(r_1)}, y_2^{(r_2)}, \dots, y_p^{(r_p)})}{\partial u} = p$$
- *the relative degrees of the p scalar output functions are all finite, and the list of relative degrees equals the list of orders of zeros at infinity.*

Proof. The sufficiency follows from the proof of Theorem 7.3. For the necessity, note that (8.4) implies that all the relative degrees are finite and condition (8.3) yields the independence of the inputs in $y_i^{(r_i)}$. ∎

8.3 Dynamic State Feedback Solution

A general solution to the above problem, consisting of a dynamic compensator, is characterized by the following theorem.

Theorem 8.2. *The noninteracting control problem is solvable via a regular dynamic compensator of the form (8.2) if and only if the system Σ is right-invertible.*

Proof. Necessity is obvious, and sufficiency may be proved by deriving a standard dynamic compensator, a so-called Singh compensator, from the inversion algorithm. It yields $y_i^{(n_{ie})} = v_i$ for any $i = 1, \dots, p$.
These statements follow from the consideration of the inversion equations (5.9), where the derivatives of the output range from $y_i^{(n'_i)}$ to $y_i^{(n_{ie})}$. If $n_{ie} \neq n'_i$, set $z_{i1} = y_i^{(n'_i)}, \dots, z_{i,n_{ie}-n'_i} = y_i^{(n_{ie}-1)}$, and solve the equations in u:

$$y_i^{(n_{ie})} = v_i$$

whose solution is $u = \alpha(x, z) + \beta(x, z)v$.
Finally, the dynamic compensator, which solves the problem, reads

$$\begin{cases} \dot{z}_{i1} = z_{i2} \\ \quad\vdots \\ \dot{z}_{i,n_{ie}-n'_i} = v_i, \text{ for } i = 1, \ldots, p \\ u = \alpha(x, z) + \beta(x, z)v \end{cases}$$

∎

8.4 Noninteracting Control via Quasi-static State Feedback

Problem Statement

Given the system Σ (8.1), find, if possible, a quasi-static state feedback $u = \alpha(x, v, \dot{v}, \ldots)$ such that, for every $i = 1, \ldots, m$,

i)
$$\mathrm{d}y_i^{(k)} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}v_i, \ldots, \mathrm{d}v_i^{(k)}\}, k \geq 0 \tag{8.5}$$

ii)
$$\mathrm{d}y_i^{(n)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \tag{8.6}$$

Recall that a quasi-static state feedback is invertible by Definition 6.3.

Theorem 8.3. *The noninteracting control problem is solvable via quasi-static state feedback if and only if the system is right-invertible.*

Proof. Necessity follows from condition (8.6); the decoupled system is necessarily right-invertible. The sufficiency follows from the construction of the canonical form (6.17). ∎

Example 8.4. Consider the unicycle in Example 3.20 with the two outputs $y_1 = x_1$ and $y_2 = x_2$. The condition in Theorem 8.1 is not satisfied since $\mathrm{rank}\, \partial(\dot{y}_1, \dot{y}_2)/\partial u = 1$. The system is right-invertible, however, and it can be decoupled by a quasi-static state feedback. Such a solution is obtained following the procedure described in Section 6.4, derived from the inversion algorithm:

$$u_1 = z/\cos x_3$$
$$u_2 = \cos^2 x_3(v_2 - v_1 \tan x_3)/z$$

The closed-loop system reduces to two decoupled linear systems, a first order system $\dot{y}_1 = v_1$ and a second order one $\ddot{y}_2 = v_2$. A standard dynamic decoupling compensator is

$$\begin{cases} \dot{z} = v_1 \\ u_1 = z/\cos x_3 \\ u_2 = \cos^2 x_3 \dfrac{v_2 - v_1 \tan x_3}{z} \end{cases}$$

Quasi-static state feedbacks are viewed herein as a mathematical tool that describes standard decoupling dynamic compensators acting on extended state spaces. The main benefit in using quasi-static state feedbacks comes from the fact that they define a group of transformations, whereas the class of regular dynamic compensators does not.

Problem

8.1. Consider again the unicycle in Example 3.20 described by

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

Compute the dynamic state feedback and a quasi-static state feedback that solve the noninteracting control problem.

9

Input-state Linearization

The control strategy employed in dealing with the input-output linearization problem has the effect, when a solution to the problem exists, of fully linearizing the state-space equations of the original system if it is a single-output system with relative degree equal to n. More precisely, one obtains in that case a compensated system of the form (7.1) in which dim $\xi_1 = n$ and dim $\xi_2 = 0$. The same holds true for a multioutput system whose decoupling matrix is square and invertible, *i.e.*, condition (7.3) is satisfied and the sum of the relative degrees of the output functions equals n.
The input/state linearization problem we consider in this chapter consists of searching for output functions that fulfill the above conditions. This issue is of major importance whenever the input-output linearization technique yields a closed-loop system that contains a possibly unstable unobservable subsystem. The presence of unstable internal dynamics disqualifies the input-output linearization scheme. The approach leading to the input-state linearization, on the other hand, allows one to master internal stability, although it may not produce a linear input-output relation.
Formally, the problem is stated as follows.

9.1 Input-state Linearization Problem Statement

Given the system Σ,

$$\dot{x} = f(x) + g(x)u \tag{9.1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, and the entries of f, g are meromorphic functions, find, if possible, a regular dynamic compensator

$$\begin{cases} \dot{\xi} = M(x,\xi) + N(x,\xi)v \\ u = \alpha(x,\xi) + \beta(x,\xi)v \end{cases} \tag{9.2}$$

where $\xi \in \mathbb{R}^q$ for some integer q and a state transformation $z = \Phi(x,\xi)$ where Φ is a local diffeomorphism from $\mathbb{R}^{n+q}$ to $\mathbb{R}^{n+q}$ at almost any point of $\mathbb{R}^{n+q}$, such that the closed-loop system reads

$$\dot{z} = Az + Bv$$

where the pair (A, B) is controllable.

9.2 Static State Feedback Solution

Here, we first consider the input/state linearization problem by restricting the class of solutions considered to that of regular static state feedbacks, that is, compensators of the form (9.2) with $q = 0$. It is not restrictive in formulating the above problem to look for solutions that yield the pair (A, B) in Brunovsky canonical form.
Solving the input-state linearization problem, then, consists of searching for the largest independent Brunovsky blocks, or for the largest independent strings of integrators, or equivalently for a set of some functions in $\mathcal{K}$ that have the largest relative degree and are controlled by independent inputs. These functions are candidates for playing the role of *output functions* in the input/output linearization problem. The input/state linearization problem is then solvable by regular static state feedback if and only if the sum of the associated relative degrees is n, the dimension of the state space. These ideas are formalized in the sequel.

Theorem 9.1. *There exists a static state feedback that solves the input-state linearization problem for Σ if and only if*

(i) $\mathcal{H}_\infty = 0$ *and*
(ii) $\mathcal{H}_k$ *is closed for any* $k \geq 1$.

Condition (i) is an accessibility condition which is obviously necessary since Σ has to be transformed into a linear controllable system. Condition (ii) is an integrability condition that implies that the controllability indices of Σ are the controllability indices of the resulting closed-loop linear system.

Proof of Theorem 9.1. The conditions of Theorem 9.1 are clearly necessary. For sufficiency, let $k^* = max\{k \geq 0 | \mathcal{H}_k \neq 0\}$, $s = \dim \mathcal{H}_{k^*}$, and $\{\mathrm{d}\varphi_1, ..., \mathrm{d}\varphi_s\}$ be a basis for $\mathcal{H}_{k^*}$. Suppose that

$$\operatorname{rank} \frac{\partial(\varphi_1^{(k^*)}, ..., \varphi_s^{(k^*)})}{\partial u} < s \tag{9.3}$$

Then there exist α_i, $i = 1, ..., s$ which are not all zero and such that

$$\sum_{i=1}^{s} \alpha_i \frac{\partial \varphi_i^{(k^*)}}{\partial u} = 0$$

Let $\omega = \sum_{i=1}^{s} \alpha_i \mathrm{d}\varphi_i$ and compute $\omega^{(k^*)}$ which belongs to $\mathcal{X}$. Thus, ω is nonzero and belongs to $\mathcal{H}_{k^*+1}$. From this contradiction, we conclude that

(9.3) does not hold and $s = \dim \frac{\mathcal{H}_{k^*}^{(k^*)} + \mathcal{X}}{\mathcal{X}}$.
Since $\mathcal{H}_{k^*-1} \supset \mathcal{H}_{k^*} + \dot{\mathcal{H}}_{k^*}$, denote

$$\sigma := \dim \frac{\mathcal{H}_{k^*-1}}{\mathcal{H}_{k^*} + \dot{\mathcal{H}}_{k^*}}$$

and let $\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_s, \mathrm{d}\dot{\varphi}_1, \ldots, \mathrm{d}\dot{\varphi}_{d_s}; \mathrm{d}\psi_1, \ldots, \mathrm{d}\psi_\sigma\}$ be a basis for $\mathcal{H}_{k^*-1}$.
Suppose that

$$\mathrm{rank} \frac{\partial(\varphi_1^{(k^*)}, ..., \varphi_s^{(k^*)}, \psi_1^{(k^*-1)}, ..., \psi_\sigma^{(k^*-1)})}{\partial u} < s + \sigma \tag{9.4}$$

Then there exist α_i, $i = 1, ..., s$ and β_j, $j = 1, ..., \sigma$, where the β_j's are not all zero, such that

$$\sum_{i=1}^{s} \alpha_i \frac{\partial \varphi_i^{(k^*)}}{\partial u} + \sum_{j=1}^{\sigma} \beta_j \frac{\partial \psi_j^{(k^*-1)}}{\partial u} = 0$$

Let $\omega = \sum_{i=1}^{s} \alpha_i \mathrm{d}\dot{\varphi}_i + \sum_{j=1}^{\sigma} \beta_j \mathrm{d}\psi_j$. Thus, ω is nonzero and $\omega^{(k^*-1)} \in \mathcal{X}$. It yields $\omega \in \mathcal{H}_{k^*}$. This stands in contradiction to the fact that the β_j's are not all zero, and (9.4) does not hold. By induction, it is possible to write a basis for $\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$ as $\{\mathrm{d}\varphi_i^{(j)}, i = 1, \ldots, m, j = 0, \ldots, r_i - 1\}$ where r_i denotes the relative degree of φ_i. Input-output linearization of the *outputs* $\{\varphi_i, i = 1, \ldots, m\}$ fully linearizes the state equation in the coordinates $\{\varphi_i^{(j)}, i = 1, \ldots, m, j = 0, \ldots, r_i - 1\}$. ∎

9.2.1 Dynamic State Feedback Solution

If static state feedback solutions do not exist, it becomes interesting to look for possible solutions in the class of dynamic state feedbacks. Once again, it is not restrictive in formulating the above problem to look for solutions that yield the pair (A, B) in Brunovsky canonical form.
Solving the dynamic input-state linearization problem consists of searching for some functions in $\mathcal{K}$ that have the largest structure at infinity. These functions are candidates for the role of *output functions* in the input/output linearization problem. The input/state linearization problem is then solvable by regular dynamic state feedback if and only if the sum of the associated orders of zeros at infinity is equal to n, the dimension of the state space. Equivalently, these linearizing output functions define a system without zero dynamics, in the sense of Section 5.6. Following the ideas described in [89], we note that a solution of the problem considered may be derived by using the $\mathcal{H}_k$ spaces defined in Section 3.5. Since the closed-loop system has to be fully controllable, a necessary condition is $\mathcal{H}_\infty = 0$. Then a canonical basis for the $\mathcal{H}_k$'s can be constructed as follows. Let $\{\omega_{k^*}\}$ be a basis of $\mathcal{H}_{k^*}$:

$$\mathcal{H}_{k^*} = \mathrm{span}_{\mathcal{K}}\{\omega_{k^*}\}$$

Let $\{\omega_{k^*-1}\}$ be such that $\{\omega_{k^*}, \dot{\omega}_{k^*}, \omega_{k^*-1}\}$ is a basis of $\mathcal{H}_{k^*-1}$:

$$\mathcal{H}_{k^*-1} = \mathrm{span}_{\mathcal{K}}\{\omega_{k^*}, \dot{\omega}_{k^*}, \omega_{k^*}\}$$

More generally, let $\{\omega_k\}$ be such that

$$\mathcal{H}_k = (\mathcal{H}_{k+1} + \dot{\mathcal{H}}_{k+1}) \oplus \mathrm{span}_{\mathcal{K}}\{\omega_k\}$$

for $k = 1, ..., k^*$. A sufficient condition for the existence of general solutions of the input/state linearization problem is as follows.

Theorem 9.2. *If $\{\omega_1, ..., \omega_{k^*}\}$ is integrable, then the dynamic state feedback linearization problem is solvable.*

Example 9.3. Consider the unicycle in Example 3.20. From the computation

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}\\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{(\sin x_3)\mathrm{d}x_1 - (\cos x_3)\mathrm{d}x_2\}\\ \mathcal{H}_3 &= 0\end{aligned}$$

$k^* = 2$, $\omega_2 = (\sin x_3)\mathrm{d}x_1 - (\cos x_3)\mathrm{d}x_2$.
Since $\dot{\omega}_2 = u_2 \cos x_3 \mathrm{d}x_1 + u_2 \sin x_3 \mathrm{d}x_2 - u_1 \mathrm{d}x_3$, one may pick $\omega_1 = \mathrm{d}x_1$, so that

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\omega_2, \dot{\omega}_2, \omega_1\}\\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\omega_2\}\end{aligned}$$

Finally, $\mathrm{span}_{\mathcal{K}}\{\omega_2, \omega_1\}$ is integrable and equals $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}y_2\}$ where $y_1 = x_1$ and $y_2 = x_2$ is a set of linearizing outputs.

Proof (Proof of Theorem 9.2.). Let $\{\mathrm{d}y_1, ..., \mathrm{d}y_m\}$ be a basis for $\mathrm{span}_{\mathcal{K}}\{\omega_1, ..., \omega_{k^*}\}$. By construction, the sum of the orders of zeros at infinity of $\{\omega_1, ..., \omega_{k^*}\}$ equals n. This sum cannot decrease by a change of basis, so it equals also the sum of the orders of zeros at infinity of $\{\mathrm{d}y_1, ..., \mathrm{d}y_m\}$. Consequently, $\{\mathrm{d}y_1, ..., \mathrm{d}y_m\}$ defines a set of output functions without any zero dynamics, and it embodies a solution of the dynamic feedback linearization problem. ∎

9.3 Partial Linearization

When the input-state linearization problem has no solution, it is interesting to investigate a more general problem, first stated and solved by Marino [112]. Actually, one may try to find the largest linearizable subsystem of a given system Σ. The closed-loop system obtained by linearizing such a subsystem will read

$$\begin{cases}\dot{z}_1 = A_1 z_1 + B_1 v_1\\ \dot{z}_2 = f_2(z) + g_2(z)\begin{pmatrix} v_1\\ v_2\end{pmatrix}\end{cases} \tag{9.5}$$

where $\dim z_1 + \dim z_2 = n$ and $\dim v_1 + \dim v_2 = m$. Necessarily, the dimension of the largest Brunovsky block of the pair (A_1, B_1) is less than or equal to k^*. Let $\overline{\mathcal{H}}_k$ denote the largest closed subspace contained in $\mathcal{H}_k$. Consequently, a sequence

$$\overline{\mathcal{H}}_1 \supset \overline{\mathcal{H}}_2 \supset \cdots \supset \overline{\mathcal{H}}_k \supset \cdots$$

is associated with the sequence $\{\mathcal{H}_k\}$. By definition, $\mathcal{H}_1 = \overline{\mathcal{H}}_1$ and $\mathcal{H}_k \supset \overline{\mathcal{H}}_k$. By construction of the $\mathcal{H}_k$'s, the following also holds:

$$\mathcal{H}_k \supset \mathcal{H}_{k+1} + \dot{\mathcal{H}}_{k+1}$$

and

$$\overline{\mathcal{H}}_k \supset \overline{\mathcal{H}}_{k+1} + \dot{\overline{\mathcal{H}}}_{k+1}$$

This allows us to define

$$n_1 := \dim \frac{\dot{\overline{\mathcal{H}}}_1 + \mathcal{X}}{\overline{\mathcal{H}}_2^{(2)} + \mathcal{X}}$$

and more generally, for $k = 2, ..., n$,

$$n_k := \dim \frac{\overline{\mathcal{H}}_k^{(k)} + \mathcal{X}}{\overline{\mathcal{H}}_{k+1}^{(k+1)} + \mathcal{X}} \tag{9.6}$$

First, consider the special case of single-input, accessible systems.

Theorem 9.4. *If $m = 1$, then the largest linearizable subsystem has dimension s where*

$$s = \max\{k \geq 1 \mid \overline{\mathcal{H}}_k \neq 0\}$$

Since $\mathcal{H}_\infty$, s always exists and $1 \leq s \leq n$.

Proof. Let z_{11} be the first component of z_1 in (9.5). Denote $\zeta := \dim z_1$. Assume without loss of generality that (A_1, B_1) is in Brunovsky canonical form. Necessarily, $\mathrm{d}z_{11} \in \mathcal{H}_\zeta$ and $\mathcal{H}_\zeta \supset \mathcal{H}_s$. This shows that in any partial linearization, $\zeta \leq s$.

It now remains to show that there exists a solution that linearizes a subsystem of order s. Since $m = 1$, $n_s = 1$ and $n_i = 0$ for any $i = 1, ..., n$, $i \neq s$. Pick $\mathrm{d}z_{11} \in \mathcal{H}_s$, apply standard input-output linearization, and the result follows. ∎

In the multiinput case, one defines similarly m dummy outputs that can be linearized and decoupled following the standard procedure.

Theorem 9.5. *Consider an accessible system; then, the largest linearizable subsystem has dimension*

$$n_1 + 2n_2 + ... + sn_s.$$

Proof. Pick $d\varphi_{11}, ..., d\varphi_{1,n_1}$ in $\overline{\mathcal{H}}_1$ and more generally $d\varphi_{k,1}, ..., d\varphi_{k,n_k}$ in $\overline{\mathcal{H}}_k$, for $k = 2, ..., s$, such that

$$\text{rank}\frac{\partial(\dot{\varphi}_{11}, ..., \dot{\varphi}_{1,n_1}, ..., \varphi_{s,1}^{(s)}, ..., \varphi_{s,n_s}^{(s)})}{\partial u} = n_1 + ... + n_s$$

Standard decoupling of these dummy outputs yields a linear subsystem that consists of n_1 blocks of dimension 1 and more generally of n_i blocks of dimension i, for $i = 2, ..., s$. It remains to show that any partial linearization has a lower dimension. Consider the partially linearized system (9.5), and assume without loss of generality that (A_1, B_1) is in Brunovsky canonical form. Let p_i denote the number of Brunovsky blocks of order i, for $i \geq 1$. Necessarily, $p_i = 0$ for $i > s$ and $p_s \leq n_s$.
Investigation of further steps yields

$$p_s + p_{s-1} \leq n_s + n_{s-1}$$

and more generally,

$$\sum_{i=k}^{i=s} p_i \leq \sum_{i=k}^{i=s} n_i$$

for any k, $1 \leq k \leq s$. This yields the claimed result. ∎

Example 9.6. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_3 & 0 \\ x_4 & 0 \\ \vdots & \vdots \\ x_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{9.7}$$

Then, compute

$$\mathcal{H}_2 = \text{span}_{\mathcal{K}}\{x_3 dx_1 - dx_2, \ldots, x_n dx_1 - dx_{n-1}\}$$

and more generally, for $2 \leq k \leq n - 1$,

$$\mathcal{H}_k = \text{span}_{\mathcal{K}}\{x_3 dx_1 - dx_2, \ldots, x_{n-k} dx_1 - dx_{n-k+1}\},$$

$$\mathcal{H}_n = \mathcal{H}_\infty = 0$$

Consequently, $n_1 = 2$, $n_i = 0$ for any $i \geq 2$. The integrability conditions (ii) in Theorem 9.1 are not satisfied. Theorem 9.5 yields that the largest feedback linearizable subsystem has dimension 2 and consists of two one-dimensional subsystems.
Note that system (9.7) is accessible and that any nonconstant function of the state has relative degree 1 only.

Theorem 9.1 is a special case of Theorem 9.5 and is equivalent to

Corollary 9.7. *System (9.1) can be fully linearized via static state feedback if and only if*

$$\sum_{i=1}^{i=m} i n_i = n$$

The integers n_k may alternatively be computed as

Corollary 9.8. *For $k = 1, \ldots, n$,*

$$n_k = \dim \frac{\overline{\mathcal{H}}_k}{\overline{\mathcal{H}}_{k+1} + \dot{\overline{\mathcal{H}}}_{k+1}} \tag{9.8}$$

Proof. Choose bases of the following spaces as

$$\overline{\mathcal{H}}_{k+1} = \operatorname{span}\{\mathrm{d}\varphi_{k+1}\}$$

$$\begin{aligned} \overline{\mathcal{H}}_k &= \operatorname{span}\{\mathrm{d}\varphi_{k+1}, \mathrm{d}\dot{\varphi}_{k+1}, \mathrm{d}\psi_k\} \\ &= (\overline{\mathcal{H}}_{k+1} + \dot{\overline{\mathcal{H}}}_{k+1}) \oplus \operatorname{span}\{\mathrm{d}\psi_k\} \end{aligned}$$

Thus, in (9.8), $n_k = \dim \operatorname{span}\{\mathrm{d}\psi_k\}$. Now compute

$$\begin{aligned} \overline{\mathcal{H}}_k^{(k)} + \mathcal{X} &= \operatorname{span}\{\mathrm{d}\varphi_{k+1}^{(k)}, \mathrm{d}\varphi_{k+1}^{(k+1)}, \mathrm{d}\psi_k^{(k)}, \mathrm{d}x\} \\ &= \operatorname{span}\{\mathrm{d}\varphi_{k+1}^{(k+1)}, \mathrm{d}\psi_k^{(k)}, \mathrm{d}x\} \end{aligned}$$

and

$$\overline{\mathcal{H}}_{k+1}^{(k+1)} + \mathcal{X} = \operatorname{span}\{\mathrm{d}\varphi_{k+1}^{(k+1)}, \mathrm{d}x\}$$

Returning to Definition (9.6), $n_k = \dim \operatorname{span}\{\mathrm{d}\psi_k^{(k)}\}$, and the result follows. ∎

Example 9.9. Consider the unicycle in Example 3.20 with the slight modification in the inputs: let the velocity u_1 be a fourth state, and let the acceleration $\dot{u}_1$ be a new input, denoted v_1. The angular velocity u_2 remains the second controlled input and is denoted v_2. The system's description then becomes

$$\dot{x} = \begin{pmatrix} x_4 \cos x_3 \\ x_4 \sin x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{9.9}$$

Compute, $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$, $\mathcal{H}_3 = 0$, and

$$\mathcal{H}_2^{(2)} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}(v_1 \cos x_3 - v_2 x_4 \sin x_3), \mathrm{d}(v_1 \sin x_3 + v_2 x_4 \cos x_3)\}$$

System (9.9) is fully linearizable by regular static state feedback obtained by solving the following equations in v:

$$v_1 \cos x_3 - v_2 x_4 \sin x_3 = w_1$$

$$v_1 \sin x_3 + v_2 x_4 \cos x_3 = w_2$$

which yield

$$v_1 = w_1 \cos x_3 + w_2 \sin x_3$$

$$v_2 = (w_2 \cos x_3 - w_1 \sin x_3)/x_4.$$

The linearizing state coordinates are $z_1 = x_1$, $z_2 = x_4 \cos x_3$, $z_3 = x_2$, and $z_4 = x_4 \sin x_3$. This solution is equivalently obtained when considering x_1 and x_2 as outputs and applying the input-output linearization technique.

Problem

9.1. Consider the realization of the ball and beam example, obtained from Exercise 2.2.

1. Check the full linearization of the system.
2. Compute the largest linearizable subsystem.

10

Disturbance Decoupling

The disturbance decoupling problem is basic in control theory, and its study has fostered the development of the so-called geometric approach in linear systems theory [160], [6] as well as in nonlinear systems theory [86, 126]. The solution of that problem by an invertible state feedback is well established by means of standard geometric tools in [86, 126] and it will not be considered here. We will instead concentrate on a more general situation, looking for a quasi-static state feedback that achieves decoupling of the disturbance from the output.
The idea of the solution is that of using feedback to make the output independent from those state components whose evolution is influenced by the disturbance. In other words, this means to make unobservable a suitable subspace of the state space in the compensated system. This strategy is clarified by the following simple example.

Example 10.1.

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= w \\ y &= x_1 \end{aligned} \tag{10.1}$$

where u is the control and w the disturbance. Through x_2, the disturbance w affects the output y:

$$\ddot{y} = w + \dot{u}.$$

The invertible (static) state feedback $u = -x_2 + v$ renders x_2 unobservable and decouples the disturbance w from the output y in the closed-loop system:

$$y^{(k)} = v^{(k-1)},\ k \geq 1.$$

The general solution of the problem achieves the same goal; it renders unobservable (under feedback) the largest subspace of the state space, and the disturbance is rejected from the output if the disturbance affects only the largest possible subspace. This scheme is displayed in Figure 10.1.

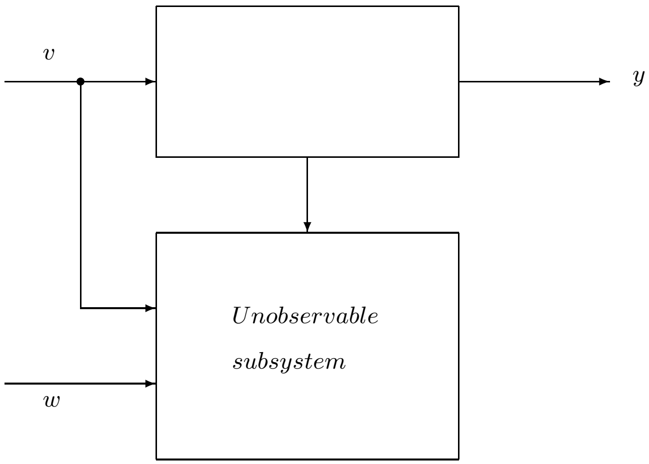

Fig. 10.1. Disturbance decoupled system

10.1. Disturbance Decouling Problem
Given the system

$$\Sigma = \begin{cases} \dot{x} = f(x) + g(x)u + p(x)w \\ y = h(x) \end{cases} \tag{10.2}$$

where the state $x \in I\!R^n$, the input $u \in I\!R^m$, the output $y \in I\!R^p$, the disturbance $w \in I\!R^q$, and the entries of f, g, p, h are meromorphic functions, find, if possible, a regular dynamic compensator

$$\begin{cases} \dot{z} = F(x,z) + G(x,z)v \; z \in I\!R^s \\ u = \alpha(x,z) + \beta(x,z)v \end{cases}$$

such that

$$\mathrm{d}y^{(i)} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}z, \mathrm{d}v, \mathrm{d}\dot{v}, \ldots\} \text{ for any } i \in I\!N \tag{10.3}$$

10.1 Solution of the Disturbance Decoupling Problem

The solution of the above problem is related to the structure of the smallest subspace that is observable under transformations induced by regular dynamic compensators. Recall from Chapter 4 that the observable space of system (10.2) with $w = 0$ is given by $\mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$. The subspace $\mathcal{X} \cap \mathcal{Y}$ of the observable space turns out to be the smallest observable subspace under transformations induced by regular dynamic compensators. It therefore gives the key for solving the disturbance decoupling problem, as described in the following theorem.

Theorem 10.2. *The disturbance decoupling problem is solvable if and only if $p(x)$ is orthogonal to the subspace $\mathcal{X} \cap \mathcal{Y}$.*

Before proving Theorem 10.2, let us give an illustrative example, taken from [80].

Example 10.3. Consider the system Σ described by the following equations

$$\begin{cases} \dot{x}_1 = & x_2 u_1 \\ \dot{x}_2 = & x_5 \\ \dot{x}_3 = & x_2 + x_4 + x_4 u_1 \\ \dot{x}_4 = & u_2 \\ \dot{x}_5 = & x_1 u_1 + w \\ y_1 = & x_1 \\ y_2 = & x_3 \end{cases} \tag{10.4}$$

From the structure algorithm, one computes

$$\mathcal{X} \cap \mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_3, (1 - \frac{x_4 \dot{y}_1}{x_2^2})\mathrm{d}x_2 + (1 + \frac{\dot{y}_1}{x_2})\mathrm{d}x_4\}$$

Since $p(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, the condition in Theorem 10.2 is satisfied. Solving for u the equations

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \ddot{y}_2 &= v_2 \end{aligned} \tag{10.5}$$

one gets

$$\begin{aligned} u_1 &= v_1/x_2 \\ u_2 &= \frac{x_2}{x_2 + v_1}\left(v_2 - x_5 + \frac{x_5 x_4}{x_2^2} v_1 - \frac{x_4}{x_2}\dot{v}_1\right) \end{aligned} \tag{10.6}$$

and one can construct the regular dynamic compensator

$$\begin{cases} \dot{z} = w_1 \\ u_1 = z/x_2 \\ u_2 = \dfrac{x_2}{x_2 + z}\left(w_2 - x_5 + \dfrac{x_5 x_4 z}{x_2^2} - \dfrac{x_4}{x_2} w_1\right) \end{cases}$$

Hence, in the compensated system, $\ddot{y}_1(t) = v_1(t)$ and $\ddot{y}_2(t) = v_2(t)$ as desired.

Proof (Proof of Theorem 10.2). Necessity follows from the fact that in the compensated system one wants to have $p(x) \perp (\mathcal{X} \cap \mathcal{Y})$ and the space $\mathcal{X} \cap \mathcal{Y}$ does not change under the action of the compensator.
Conversely, if $p(x) \perp (\mathcal{X} \cap \mathcal{Y})$, the disturbance input w does not appear in the equations of the form (5.9) obtained by applying the Structure Algorithm to a system Σ of the form (10.2). As illustrated in the above example, this allows us to construct a regular dynamic compensator that guarantees (10.3). ∎

11

Model Matching

In the nonlinear framework, the model matching problem was considered in [43] and, in the case of a linear model, in [39, 84]. Some further contributions are in [130]. The formulation of the model matching problem that we give in the following differs slightly from that of [43]. However, our approach provides a condition for the solution of the problem which is at the same time necessary and sufficient, whereas the conditions given in [43] are either necessary or sufficient.

11.1 A Special Form of the Inversion Algorithm

Given the system Σ, we may consider its input u as divided into two subsets $u = (v, w)$, where v is viewed as a set of controls and w as a set of parameters. In this case, we apply the following algorithm to Σ.

Algorithm 11.1

Step 1.
Calculate

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g_v(x)v + g_w(x)w =: f_1(x, w) + g_1(x)v$$

and set $G_1(x) := g_1(x)$ *and* $s_1 := \mathrm{rank} G_1(x)$. *Permute, if necessary, the rows of the output so that the first* s_1 *rows of* $G_1(x)$ *are linearly independent, and decompose* $\dot{y}$ *as*

$$\dot{y} = \begin{pmatrix} \tilde{y}_1 \\ \hat{y}_1 \end{pmatrix} \tag{11.1}$$

where $\dim \tilde{y}_1 = s_1 =: \rho_{1v}$. *Then, eliminating* v *in the last rows, write*

$$\begin{pmatrix} \tilde{y}_1 \\ \hat{y}_1 \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(x, w) + \tilde{g}_1(x)v \\ \hat{y}(x, w, \tilde{y}_1) \end{pmatrix} \tag{11.2}$$

and set $\tilde{G}_1(x) =: \tilde{g}_1(x)$.

Step k+1.
Suppose that from steps 1 through k,

$$\begin{aligned}
\tilde{y}_1 &= \tilde{f}_1(x,w) + \tilde{g}_1(x)v \\
&\vdots \\
\tilde{y}_k &= \tilde{f}_k(x,w,\dots,w^{(k-1)},\tilde{y}_1,\dots,\tilde{y}_1^{(k-1)},\dots,\tilde{y}_{k-1},\dot{\tilde{y}}_{k-1}) \\
&\quad +\tilde{g}_k(x,w,\dots,w^{(k-2)},\tilde{y}_1,\dots,\tilde{y}_1^{(k-2)},\dots,\tilde{y}_{k-1})v \\
\hat{y}_k &= \hat{y}_k(x,w,\dots,w^{(k-1)},\tilde{y}_1,\dots,\tilde{y}_1^{(k-1)},\dots,\tilde{y}_k)
\end{aligned}$$

where

$$\tilde{G}_k = \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_k \end{pmatrix}$$

has full rank s_k. Then

$$\begin{aligned}
\dot{\hat{y}}_k &= f_{k+1}(x,w,\dots,w^{(k)},\tilde{y}_1,\dots,\tilde{y}_1^{(k)},\dots,\tilde{y}_k,\dot{\tilde{y}}_k) \\
&\quad +g_{k+1}(x,w,\dots,w^{(k-1)},\tilde{y}_1,\dots,\tilde{y}_1^{(k-1)},\dots,\tilde{y}_k)v
\end{aligned}$$

Define $G_{k+1} := \begin{pmatrix} \tilde{G}_k \\ g_{k+1} \end{pmatrix}$ *and* $s_{k+1} := \mathrm{rank} G_{k+1}(x)$. *Decompose* $\dot{\hat{y}}_k$ *as*

$$\dot{\hat{y}}_k = \begin{pmatrix} \tilde{y}_{k+1} \\ \hat{y}_{k+1} \end{pmatrix}$$

where $\dim \tilde{y}_{k+1} = s_{k+1} - s_k =: \rho_{(k+1)v}$. *Then, eliminating v in the last rows, write*

$$\begin{aligned}
\tilde{y}_{k+1} &= \tilde{f}_{k+1}(x,w,\dots,w^{(k)},\tilde{y}_1,\dots,\tilde{y}_1^{(k)},\dots,\tilde{y}_k,\dot{\tilde{y}}_k) \\
&\quad +\tilde{g}_{k+1}(x,w,\dots,w^{(k-1)},\tilde{y}_1,\dots,\tilde{y}_1^{(k-1)},\dots,\tilde{y}_k)v \\
\hat{y}_{k+1} &= \hat{y}_{k+1}(x,w,\dots,w^{(k)},\tilde{y}_1,\dots,\tilde{y}_1^{(k)},\dots,\tilde{y}_k,\dot{\tilde{y}}_{k+1})
\end{aligned}$$

and set

$$\tilde{G}_{k+1} := \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k+1} \end{pmatrix}$$

End of the algorithm.

Algorithm 11.1 performs the inversion of Σ, viewed as a system depending on the parameter w, with respect to the input v when $\rho_v := s_n$ equals the dimension of v. When w is empty, Algorithm 11.1 reduces to the usual Singh's inversion algorithm. The indices σ_i, s_i, and ρ_i contain the same information, and each of them could be used in the following. We choose to state the next results in terms of ρ_i, which have a direct interpretation as numbers of zeros at infinity of order i (see [119]), although the other indices are often used in proofs and calculations.

Lemma 11.2. *Let the systems*

$$T = \begin{cases} \dot{x} &= f(x) + g(x)u \\ y_T &= h(x) + h'(x)u \end{cases} \tag{11.3}$$

and

$$G = \begin{cases} \dot{z} &= f_G(z) + g_G(z)v \\ y_G &= h_G(z) \end{cases} \tag{11.4}$$

with outputs of the same dimension, be given, and let (GT) denote the composite system

$$(GT) = \begin{cases} \dot{x} &= f(x) + g(x)u \\ \dot{z} &= f_G(z) + g_G(z)v \\ y_{GT} &= h(x) - h_G(z) + h'(x)u \end{cases} \tag{11.5}$$

Then, $\rho_{iv}(GT) = \rho_i(G)$ for all i and, in particular, $\rho_v(GT) = \rho(G)$.

Proof. Let $\mathcal{K}$' denote the field of meromorphic functions in the variables $x, z, v, \dots, v^{(N-1)}$ and the parameters $u, \dots, u^{(N)}$, where $N = \dim x + \dim z$. We denote by $\mathcal{E}_i^{GT}$ the vector space spanned over $\mathcal{K}$' by

$$\{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\dot{y}_{GT}, \dots, \mathrm{d}y_{GT}^{(i)}\}$$

Note that to consider $u, \dots, u^{(N)}$ as parameters instead of variables means that the differential $\mathrm{d}(\cdot)$ is given by

$$\mathrm{d}(\cdot) = (\partial(\cdot)/\partial x)\mathrm{d}x + (\partial(\cdot)/\partial z)\ \mathrm{d}z + \sum_{i=0}^{N-1} (\partial(\cdot)/\partial v^{(i)})\mathrm{d}v^{(i)}$$

Following the proof given in ([42], Thm. 2.3) one can show that $\rho_{iv}(GT) = \dim_{\mathcal{K}'}\ \mathcal{E}_i^{GT}/\mathcal{E}_{i-1}^{GT}$. From this, since

$$\begin{aligned} \mathrm{d}y_{GT}^{(j)} &= \mathrm{d}y_T^{(j)} - \mathrm{d}y_G^{(j)} \\ &= \phi_j(x, u, \dots, u^{(j)})\mathrm{d}x - \mathrm{d}y_G^{(j)} \end{aligned}$$

with $\phi_j \in \mathcal{K}'$, it follows that

$$\begin{aligned} \rho_{iv}(GT) = \dim \mathrm{span}_{\mathcal{K}'}\{dx, dz, d\dot{y}_G, \dots, dy_G^{(i)}\} \\ -\dim \mathrm{span}_{\mathcal{K}'}\{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\dot{y}_G, \dots, \mathrm{d}y_G^{(i-1)}\} \end{aligned}$$

and hence

$$\rho_{iv}(GT) = \dim \mathrm{span}_{\mathcal{K}'}\{\mathrm{d}z, \mathrm{d}\dot{y}_G, \dots, \mathrm{d}y_G^{(i)}\} - \dim \mathrm{span}_{\mathcal{K}'}\{\mathrm{d}z, \mathrm{d}\dot{y}_G, \dots, \mathrm{d}y_G^{(i-1)}\}$$

Now, let $\{w_1, \dots, , w_{r_i}\} \subset \{\mathrm{d}z, \mathrm{d}\dot{y}_G, \dots, \mathrm{d}y_G^{(i)}\}$ be a basis over $\mathcal{K}'$ of $\mathcal{E}_i^{GT}$, and let $\tilde{w}$ be an element of $\{w_1, \dots, w_{r_i}\}$. We write

$$\tilde{w} = \varSigma\gamma_j(x, z, v, \dots, v^{(N-1)}, u, \dots, u^{(N)})w_j$$

with $\gamma_j \in \mathcal{K}'$ and, computing the derivatives with respect to $x, u, \ldots, u^{(N)}$,

$$\begin{aligned}
\frac{\partial \tilde{w}}{\partial x} &= \sum \frac{\partial \gamma_j}{\partial x} w_j &= 0 \\
\frac{\partial \tilde{w}}{\partial u} &= \sum \frac{\partial \gamma_j}{\partial u} w_j &= 0 \\
&\vdots \\
\frac{\partial \tilde{w}}{\partial u^{(N)}} &= \sum \frac{\partial \gamma_j}{\partial u^{(N)}} w_j &= 0
\end{aligned}$$

Therefore, $\partial\gamma_j/\partial x = 0$ and $\partial\gamma_j/\partial u = \ldots = \partial\gamma_j/\partial u^{(N)} = 0$ for all j, or, equivalently, $\gamma_j = \gamma_j(z, v, \ldots, v^{(N-1)})$. This says that $\{w_1, \ldots, w_{r_i}\}$ is a set of generators over the field $\mathcal{K}$ of meromorphic functions in the variables $(z, v, \ldots, v^{(N-1)})$ of $\text{span}_{\mathcal{K}}\{\mathrm{d}z, \mathrm{d}\dot{y}_G, \ldots, \mathrm{d}y_G^{(i)}\} = \mathcal{E}_i^G$. Moreover, since $\mathcal{K} \subset \mathcal{K}'$, $\dim_{\mathcal{K}} \mathcal{E}_i^{GT} = \dim_{\mathcal{K}} \mathcal{E}_i^G$ for all i, and the result follows. ∎

11.2 Model Matching Problem

Let us now state the model matching problem (MMP).

Problem Statement

Given a model

$$T = \begin{cases} \dot{x} &= f(x) + g(x)u \\ y_T &= h(x) \end{cases} \tag{11.6}$$

and a system G as in (11.4), find a proper compensator

$$H = \begin{cases} \dot{\xi} = f_H(\xi, z, u) \\ v = h_H(\xi, z, u) \end{cases}$$

with state space $I\!R^q$ and a map $\phi : I\!R^n \to I\!R^q$ such that, denoting by y_{GH} the output of the composite system GH, $y_T(u, x) - y_{GH}(u, \phi(x), z)$, that is, the difference between the output of the model, viewed as a function of u and of the initial state x, and the output of the composite system, viewed as a function of u and of a suitably defined initial state z and $\xi = \phi(x)$, does not depend on u.

To gain a better insight into the model matching problem that we are considering, we now state it in a generalized form (GMMP), which includes in particular the left inversion problem. Specializing such a formulation by requiring a proper compensator, we get the most interesting case from the point of view of control theory.

Problem Statement (Generalized form)

Given a model T as in (11.3) and a system G as in (11.4), find an integer $\nu \geq 0$, a possibly nonproper compensator

$$H = \begin{cases} \dot{\xi} = f_H(\xi, z, u, \dots, u^{(\nu)}) \\ v = h_H(\xi, z, u, \dots, u^{(\nu)}) \end{cases} \tag{11.7}$$

with state space $I\!R^q$, and a map $\phi : I\!R^n \to I\!R^q$ such that, denoting by y_{GH} the output of the composite system GH, $y_T(u, x) - y_{GH}(u, \phi(x), z)$, that is, the difference between the output of the model, viewed as a function of u and of the initial state x, and the output of the composite system, viewed as a function of u and of the initial states z and $\xi = \phi(x)$, does not depend on $u^{(\nu)}$.

The MMP is the special case of the GMMP for $\nu = 0$.

Remark 11.3. In the MMP the requirement that $y_T(u, x) - y_{GH}(u, \phi(x), z)$ does not depend on u amounts, in the linear case, to the equality of the transfer functions of the model and of the composite systems. From this point of view, therefore, our formulation represents the natural extension of the one currently understood for the linear model matching problem (compare with the quoted references and with [86, 84]).

We recall that a stronger formulation of the MMP, requiring the equality of y_T and y_{GH}, has been considered, only for a linear model, in [39]. Note that the problem we stated qualifies as an exact MMP, as opposed to an approximate or an asymptotic MMP that could also be considered, see, e.g., [78].

Let us consider the left inversion problem in the linear framework. The solution provided by the Silverman algorithm [144] has the form (11.7), where ν is the inherent integration order of the system [143, 127]. In the simple example given by $T = \{y_T = y$ and by

$$G = \begin{cases} \dot{z} = v \\ y = z \end{cases}$$

we obtain

$$H = G^{-1} = \begin{cases} \dot{\xi} = \dot{y} \\ v = \dot{y} \end{cases}$$

The difference between the outputs of the identity model T and of GH is $y_T - y_{GH} = y - z$, the latter depends on the input y and is independent of the first derivative $\dot{y}$.

Example 11.4. Let

$$T = \begin{cases} \dot{x} = u \\ y_T = x \end{cases}$$

and

$$G = \begin{cases} \dot{z} = v \\ y_G = z^2 \end{cases}$$

be the data of a MMP. The pair consisting of the compensator

$$H = \{v = u/2z$$

and of the empty function is a solution in the sense of Remark 11.3. For $z_0 \neq 0$,

$$y_{GH}(u, z_0) = \int_0^t u(\tau) d\tau + z_0^2$$

for all input functions $u(t)$ and for all $t > 0$ such that $\int_0^t u(\tau) d\tau + z_0^2 > 0$. Then,

$$y_T(u, x) - y_{GH}(u, z) = x - z^2$$

and

$$\mathrm{d}(y_T(u, x) - y_{GH}(u, z))^{(k)} \in \mathrm{span}_{\mathcal{K}} \{\mathrm{d}x, \mathrm{d}z\}$$

In particular, if, for example, the input is bounded by $|u(t)| \leq M$ and the initial conditions $x_0, z_0 \neq 0$, are chosen, $y_T - y_{GH}$ is independent of u over the time interval $[0, z_0^2/M)$. It may be useful to note that $v = u/2z$ is a solution of the MMP in the same way, that is, with the same limitations, in which it is a solution, in the sense of [87], of the disturbance decoupling problem with disturbance measurement described by $\dot{x} = u,\ \dot{z} = v,\ y = x - z^2$, where u is the disturbance and v is the control.

Note that taking, for instance,

$$T = \begin{cases} \dot{x} = xu \\ y_T = x \end{cases}$$

and

$$G = \begin{cases} \dot{z} = zv \\ y_G = z \end{cases}$$

contrarily to what happens in the linear case, the identity compensator

$$H = \{v = u$$

does not give a solution of the MMP.

$$y_T - y_{GH} = (x_0 - z_0)\ exp\ (\int_0^t u(\tau) d\tau)$$

is independent of u only if the initial states of the model and of the system coincide. In this case, a solution is given by the compensator

$$H = \begin{cases} \dot{\xi} = \xi u \\ v = (\xi/z)u \end{cases}$$

where $\xi(t) \in I\!R^n$, and $\phi = id$.

A structural condition under which a compensator exists and a procedure to compute it are given in the following theorem.

Theorem 11.5. *The generalized model matching problem is solvable if*

$$\rho(GT) = \rho(G) \tag{11.8}$$

where (GT) is the composite system (11.5).

Proof. Applying Algorithm 11.1 to (GT), we obtain

$$\begin{aligned}
\begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \\ \vdots \\ \tilde{Y}_N \\ \hat{Y}_N \end{pmatrix} &= \begin{pmatrix} \tilde{F}_1(x,z,u,\dot{u}) \\ \tilde{F}_2(x,z,u,\dot{u},\ddot{u},\tilde{Y}_1,\dot{\tilde{Y}}_1) \\ \vdots \\ \tilde{F}_N(x,z,u,\dots,u^{(N)},\tilde{Y}_1,\dots,\tilde{Y}_1^{(N-1)},\dots,\tilde{Y}_{N-1},\dot{\tilde{Y}}_{N-1}) \\ \hat{F}_N(x,z,u,\dots,u^{(N)},\tilde{Y}_1,\dots,\tilde{Y}_1^{(N-1)},\dots,\tilde{Y}_{N-1},\dot{\tilde{Y}}_{N-1},\tilde{Y}_N) \end{pmatrix} \\
&\quad + \begin{pmatrix} \tilde{G}_1(z) \\ \tilde{G}_2(x,z,u,\tilde{Y}_1) \\ \vdots \\ \tilde{G}_N(x,z,u,\dots,u^{(N-2)},\tilde{Y}_1,\dots,\tilde{Y}_1^{(N-2)},\tilde{Y}_{N-1}) \\ 0 \end{pmatrix} v \\
&= \begin{pmatrix} \tilde{F} \\ \hat{F}_N \end{pmatrix} + \begin{pmatrix} \tilde{G} \\ 0 \end{pmatrix} v
\end{aligned} \tag{11.9}$$

with rank $\tilde{G} = \#$ rows $\tilde{G} = \rho_v(GT)$ and where $\tilde{Y}_i$ represents a suitable subset of rows of $y_{GT}^{(i)}$, which will be useful to denote also as $y_{T.i}^{(i)} - y_{G.i}^{(i)}$. We can choose constant values

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{N-1} \\ 0 \end{pmatrix} \qquad \text{for } \tilde{Y} = \begin{pmatrix} \tilde{Y}_1^{(N-2)} \\ \vdots \\ \tilde{Y}_{N-1} \\ \tilde{Y}_N \end{pmatrix}$$

such that the generic rank of $\tilde{G}$ evaluated at Y is equal to the number of rows of $\tilde{G}$. Then, solving for v the system

$$\begin{pmatrix} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_N \end{pmatrix} = \tilde{F}_{|Y} + \tilde{G}_{|Y} v$$

obtained by replacing $\tilde{Y}$ with Y in (11.9), we get

$$v = \phi(x,z,u,\dots,u^{(N)},\tilde{Y}_1,\dots,\tilde{Y}_1^{(N-3)},Y_1,\dots,\tilde{Y}_{N-2},Y_{N-2},Y_{N-1})$$

Now, denoting by w_0 a vector of the same dimension as x and by w_i a vector of dimension $(N-i)\cdot \dim \tilde{Y}_i$, we set $\nu = N$, and we construct the compensator

$$H = \begin{cases} \dot{w}_0 = f(w_0) + g(w_0)u \\ \dot{w}_i = \begin{pmatrix} 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix} w_i + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Y_i \end{pmatrix} \qquad \text{for } 1 \leq i \leq N-2 \\ v = \phi(w_0, z, u, \ldots, u^{(N)}, w_1, \ldots, w_{N-2}, Y_{N-1}) \end{cases} \tag{11.10}$$

Letting $\phi(x) = (x, 0, \ldots, 0)$, we claim that (H, ϕ) is a solution of the GMMP. To show this, let us first note that, in (11.9), $\tilde{Y}_N$ is independent of $u^{(k)}$ for all k. By Lemma 11.2 and the rank equality (11.8), it follows that $\rho(GT) = \rho_v(GT)$ and we know, from ([42], Theorem 2.3), that the $\mathrm{d}x, \mathrm{d}z, \mathrm{d}\tilde{Y}_1, \ldots, \mathrm{d}\tilde{Y}_1^{(N-1)}, \ldots, \mathrm{d}\tilde{Y}_N$ are independent over the field $\mathcal{K}$. So, if $\partial \tilde{F}_N / \partial u^{(k)} \neq 0$, for some $k \geq 0$, $\mathrm{d}Y_N$ does not belong to $\mathrm{span}_{\mathcal{K}} \, \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\tilde{Y}_1, \ldots, \mathrm{d}\tilde{Y}_1^{(N-1)}, \ldots, \mathrm{d}\tilde{Y}_N\}$ and then $\rho(GT) > \rho_v(GT)$, contradicting the assumption.

Now let us consider the composite system (GH):

$$(GH) = \begin{cases} \dot{w} = F(w) + G(w)u \\ \dot{z} = f_G(z) + g_G(z)\phi(w, z, u, \ldots, u^{(N)}) \\ y_{GH} = h_G(z) \end{cases}$$

initialized at $\phi(x_0) = (x_0, 0, \ldots, 0)$ and the difference $y_T - y_{GH}$ between the output of the model and that of (GH). Recalling the notation $\tilde{Y}_i = y_{T.i}^{(i)} - y_{G.i}^{(i)}$, by substituting the output of H to v in (11.9) and taking derivatives,

$$\begin{aligned} & y_{T.i}^{(N-1)} - y_{GH.i}^{(N-1)} = Y_i \; for \; 1 \leq i \leq N-1 \\ & y_{T.N}^{(N)} - y_{GH.N}^{(N)} = 0 \end{aligned}$$

Therefore $\mathrm{d}(y_T - y_{GH})^{(k)} \in \mathrm{span}_{\mathcal{K}} \, \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}w, \mathrm{d}u, \ldots, \mathrm{d}u^{(N-1)}\}$ for all k. ∎

It is worthwhile to note that, although in (11.7) the compensator H is described in a very general form, the construction illustrated in the proof of Theorem 11.5 always produces a system whose state equations have the same form as those of the model. In particular, the derivatives of the input appear only in the output function $h_H(\xi, z, u, \ldots, u^{(n)})$. A structural condition under which there exists a proper compensator H, that is, one which does not depend on the derivatives of the input u, is given in the next theorem.

Theorem 11.6. *The MMP is solvable with a proper compensator H of the form*

$$H = \begin{cases} \dot{\xi} = f_H(\xi, z, u) \\ v = h_H(\xi, z, u) \end{cases}$$

if

$$\rho_i(GT) = \rho_i(G) \tag{11.11}$$

for all $i \geq 1$.

Proof. Assume that (11.11) holds, then, for all i, by Lemma 11.2, $\rho_{iv}(GT) = \rho_i(GT)$. In particular, this implies that at Step 1 of Algorithm 11.1 applied to (GT),

$$\begin{pmatrix} \tilde{Y}_1 \\ \hat{Y}_1 \end{pmatrix} \begin{pmatrix} \tilde{F}_1(x,z,u) \\ \hat{F}_1(x,z,u,\tilde{Y}_1) \end{pmatrix} + \begin{pmatrix} \tilde{G}_1(z) \\ 0 \end{pmatrix} v,$$

where $\partial \tilde{F}_1 / \partial u = 0$; otherwise $\rho_1(GT)$ would be strictly greater than $\rho_{1v}(GT)$. Repeatedly applying the same argument, we get at the last step N,

$$\begin{pmatrix} \tilde{Y} \\ \tilde{Y}_N \end{pmatrix} = \begin{pmatrix} \tilde{F}(x,z,u,\tilde{Y},\ldots,\tilde{Y}^{(N-1)}) \\ \hat{F}_N(x,z,u,\tilde{Y},\ldots,\tilde{Y}^{(N)}) \end{pmatrix} + \begin{pmatrix} \tilde{G}(x,z,\tilde{Y},\ldots,\tilde{Y}^{(N-2)}) \\ 0 \end{pmatrix} v$$

and, hence, $v = \phi(x,z,\tilde{Y},\ldots,\tilde{Y}^{(N-3)})$. Therefore the compensator obtained following the construction described in the proof of Theorem 11.5 is proper in this case. ∎

Example 11.7. **(i)** The MMP concerning the model

$$T = \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u \\ y_T = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \end{cases}$$

and the system

$$G = \begin{cases} \dot{z} = \begin{pmatrix} 0 \\ z_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} z_3 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} v \\ y_G = \begin{pmatrix} z_2 - z_3 \\ z_1 \end{pmatrix} \end{cases}$$

was considered in [43]. It was shown that the geometric necessary condition given in the same paper is not verified, although the compensator

$$H = \begin{cases} \dot{\zeta} = \zeta^2/(\zeta + z_3) + 1/(\zeta + z_3)(-\zeta \;\; 1)u, \\ v = \begin{pmatrix} \zeta \\ \zeta^2/(\zeta + z_3) \end{pmatrix} + 1/(\zeta + z_3) \begin{pmatrix} 0 & 0 \\ z_3 & 1 \end{pmatrix} u \end{cases}$$

provides a solution of the problem (see [43], Example 5.4). It can be easily checked that (11.11) is verified, that is, $\rho_i(GT) = \rho_i(G)$. Then, applying the procedure illustrated in the proof of Theorem 11.5, we get the proper compensator

$$H' = \begin{cases} \dot{\xi} = \begin{pmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u, \\ v = \begin{pmatrix} z_4 - \xi_2 \\ (z_4 - \xi_2 - u_1 z_3 - u_2)/(\xi_2 - z_3 - z_4) \end{pmatrix} \end{cases}$$

Clearly, by removing the unnecessary equations $\dot{\xi}_1 = \xi_2$, $\dot{\xi}_3 = \xi_4$, and $\dot{\xi}_4 = u_2$, we obtain another compensator, say H", that solves the problem. Now, the change of variables $\zeta = z_4 - \xi_2$ transforms H" into H.

(ii) Consider the model

$$T = \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y_T = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \end{cases}$$

and the system

$$G = \begin{cases} \dot{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_3 & 1 \end{pmatrix} v \\ y_G = \begin{pmatrix} z_1 \\ z_2 - z_3 \end{pmatrix} \end{cases}$$

for which $\rho(GT) = \rho(G)$. Note that, since

$$y_G = \begin{pmatrix} \int_0^t v_1(\tau) d\tau + z_1(0) \\ \int_0^t v_2(\tau) d\tau \;\; - exp\left(\int_0^t v_1(\tau) d\tau \right) \int_0^t exp\left(\int_0^t v_1(\sigma) d\sigma \right) v_2(\tau) d\tau \\ + z_2(0) + z_3(0)\; exp\left(\int_0^t v_1(\tau) d\tau \right) \end{pmatrix}$$

and $y_{T.1} = 0$, contrary to what happens in the linear case, it is not possible to find a compensator H such that $y_T - y_{GH} = 0$ for $u \neq 0$, also when we are allowed to choose the initial condition $z(0)$. Applying Algorithm 11.1 to (GT),

$$\tilde{Y} = \begin{pmatrix} \dot{\tilde{Y}}_1 \\ \ddot{\tilde{Y}}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u - z_3 \ddot{Y}_1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ z_3 \dot{Y}_1 & \dot{Y}_1 \end{pmatrix} v = \tilde{F} + \tilde{G} v$$

and then, in fixing constant values Y for $\tilde{Y}$, we are obliged to choose $\dot{Y}_1 \neq 0$. Taking, for instance, $Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get $v = \begin{pmatrix} -1 \\ u + z_3 \end{pmatrix}$, which represents by itself a compensator H that solves the problem.

Remark 11.8. The conditions of Theorems 11.5 and 11.6 are not necessary for the existence of solutions of the GMMP and the MMP, as pointed out by the following example, taken from [77]. Let

$$T = \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ x_3 \\ 0 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ y_T = \begin{pmatrix} x_2 \\ x_4 \\ x_1 \end{pmatrix} \end{cases}$$

and

$$G = \begin{cases} \dot{z} = \begin{pmatrix} z_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & z_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ y_G = \begin{pmatrix} z_2 \\ z_3 \\ z_1 \end{pmatrix} . \end{cases}$$

By applying Singh's Algorithm to G, we get $\rho_1(G) = \rho_2(G) = \rho_3(G) = 2$. The same procedure applied to (GT) gives

$$\begin{aligned} \dot{y}_{GT1} &= x_3 + u_1 - v_1, \\ \dot{y}_{GT2} &= x_4 - v_2, \\ \dot{y}_{GT3} &= x_2 + z_2(\dot{y}_{GT2} - x_4 - 1) \end{aligned}$$

and then,

$$\ddot{y}_{GT3} = \dot{y}_{GT1} + z_2(\dot{y}_{GT2} - x_4) - (\dot{y}_{GT1} - x_3 - u_1)(\dot{y}_{GT2} - x_4)$$

So $\rho_1(GT) = 2$, $\rho_2(GT) = 3$ and the sufficient conditions of Theorems 11.5 and 11.6 are not satisfied. However, the compensator

$$H = \begin{cases} \dot{\xi} = \begin{pmatrix} \xi_2 \\ \xi_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ v_1 = \xi_3 + u_1 \\ v_2 = 0 \end{cases}$$

and $\phi = id$ give a solution of the MMP.

From (11.9), we get the equality

$$\begin{aligned} \tilde{Y} = {} & \tilde{F}(x, z, u, \ldots, u^{(N)}, \tilde{Y}_1, \ldots, \tilde{Y}_1^{(N-1)}, \ldots, \tilde{Y}_{N-1}, \dot{\tilde{Y}}_{N-1}) \\ & + \tilde{G}_N(x, z, u, \ldots, u^{(N-2)}, \tilde{Y}_1, \ldots, \tilde{Y}_1^{(N-2)}, \ldots, \tilde{Y}_{N-1})v \end{aligned} \tag{11.12}$$

and, by differentiation of $\hat{Y}_N$, the equalities

$$\begin{aligned}\hat{Y}_N &= \hat{F}_N(x, z, u, \ldots, u^{(N)}, \tilde{Y}_1, \ldots, \tilde{Y}_1^{(N-1)}, \ldots, \tilde{Y}_{N-1}, \dot{\tilde{Y}}_{N-1}, \tilde{Y}_N),\\ \hat{Y}_N^{(n+q-N)} &= \hat{F}_{n+q}(x, z, u, \ldots, u^{(n+q)}, \tilde{Y}_1, \ldots, \tilde{Y}_1^{(n+q-1)}, \ldots, \tilde{Y}_{N-1},\\ &\quad \tilde{Y}^{(n+q-N+1)}, \tilde{Y}_N), \ldots, \tilde{Y}_N^{(n+q-N)})\end{aligned} \tag{11.13}$$

from which it can be understood that a weaker condition for the existence of solutions of the MMP is, in particular, that there exists a vector of functions

$$Y(x,z) = \begin{pmatrix} Y_1(x,z) \\ \vdots \\ Y_N(x,z) \end{pmatrix}$$

such that

i) $\partial Y^{(k)}/\partial u = 0$ for all k;
ii) substituting $Y(x,z)$ and its derivatives for

$$\tilde{Y} = \begin{pmatrix} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_N \end{pmatrix}$$

and its derivatives in $\tilde{G}$, the generic rank is equal to the number of rows;
iii) substituting $Y(x,z)$ and its derivatives for

$$\tilde{Y} = \begin{pmatrix} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_N \end{pmatrix}$$

and its derivatives in $\hat{F}_N, \ldots, \hat{F}_{n+q}, \tilde{F}$, all the coefficients of the monomials in $u, \ldots, u^{(n+q)}$, and, respectively, all the coefficients of the monomials in $\dot{u}, \ldots, u^{(n+q)}$, are zero. Such a condition is verified in the above example for $Y(x,z) = \begin{pmatrix} 0 \\ x_4 \end{pmatrix}$.

11.3 Left Factorization

It is well known [25, 70] that in the linear case (11.8) and (11.11) are necessary and sufficient conditions for solving the GMMP or the MMP, also when no feedback connection between the state of the system G and the precompensator H is allowed. In such a formulation, the linear GMMP amounts to the problem of factoring the transfer function of the model T through a possible left factor, represented by the transfer function of G. It is natural, then, from an abstract point of view, to consider also the dual problem, which consists of factoring the transfer function of T through a possible given right factor (see [24, 25, 70, 107]). In the more general context we are considering, this leads to the following formulation for what we call the left factorization problem.

11.3.1 Left Factorization Problem (LFP)

Problem Statement

Given a model T as in (11.6) and a system

$$H = \begin{cases} \dot{z} = f_H(z) + g_H(z)u \\ v = h_H(z) \end{cases} \tag{11.14}$$

find a proper compensator

$$G = \begin{cases} \dot{\xi} = f_G(\xi, v) \\ y_G = h_G(\xi, v) \end{cases}$$

with state space $I\!R^q$ and a map $\phi : I\!R^n \to I\!R^q$ such that, denoting by Y_{GH} the output of the cascade GH, $y_T(u, x_0) - y_{GH}(u, \phi(x_0), z_0) = 0$ for any initial states x_0, z_0.

Problem Statement(Generalized version): GLFP

Given a model T as in (11.3) and a system H as in (11.14), find an integer $\nu \geq 0$ and a possibly nonproper compensator

$$G = \begin{cases} \dot{\xi} = f_G(\xi, v, \ldots, v^{(n)}) \\ y_G = h_G(\xi, v, \ldots, v^{(n)}) \end{cases} \tag{11.15}$$

with state space $I\!R^q$, a map $\phi : I\!R^n \to I\!R^q$ such that, denoting by Y_{GH} the output of the cascade GH,

$$y_T - (u, x_0) - y_{GH} - (u, \phi(x_0), z_0) = 0 \tag{11.16}$$

for any initial states x_0, z_0.

Remark 11.9. The same considerations as in Remark 11.3 apply to the present situation. Therefore a solution (G, ϕ) will be one that achieves (11.16) for all initial states x_0, z_0 in an open and dense subset of the state spaces.

The first result we have in this framework is the following theorem.

Theorem 11.10. *The GLFP is solvable only if*

$$\rho \begin{pmatrix} T \\ H \end{pmatrix} = \rho(H) \tag{11.17}$$

where $\begin{pmatrix} T \\ H \end{pmatrix}$ *is the system consisting of the state and output equations of* T *and* H*.*

Proof. We start by proving the theorem under an additional technical assumption on the system H. Assume that the maximal regular controllability distribution $\mathcal{R}*_H$ of H contained in $\ker \mathrm{d}h_H$ is locally well defined, *i.e.*, that the regularity conditions of ([86], §6.4) are satisfied. Denoting by $\mathcal{G}$ the distribution spanned by $g_H(z)$, we assume that the following holds:

$$\dim(\mathcal{G} \cup \mathcal{R}^*_H) = m - \rho(H), \tag{11.18}$$

Now, let the regular feedback $u = \alpha(z) + \beta(z)w$ be a "friend" of $\mathcal{R}^*_H$ and let us denote by $\begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix}$ the system obtained by compensating the system $\begin{pmatrix} T \\ H \end{pmatrix}$ with $u = \alpha(z) + \beta(z)w$. By (11.18), the action of the feedback $u = \alpha(z) + \beta(z)w$ transforms H into the system $\tilde{H}$ which, up to a change in coordinates, is of the form [86] $\dot{z}_1 = f_1(z_1) + g_1(z_1)w_1,\ \dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2)w,\ v = h_{\tilde{H}}(z_1)$, where $w = (\bar{w}_1, \bar{w}_2),\ \bar{w}_1 = (w_1, \dots, w_\rho)$, and $\rho = \rho(H)$. Hence,

$$\frac{\partial v^{(k)}(w, z_0)}{\partial w_i} = 0 \tag{11.19}$$

for all $i \geq \rho + 1$ and for all k. Moreover, if (G, ϕ) is a solution of the GLFP, the output trajectory $\tilde{Y}(w, x_0, \phi(x_0), z_0)$ of the cascade composition between $\begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix}$ and the system

$$\tilde{G} = \begin{cases} \dot{\xi} = f_G(\xi, v, \dots, v^{(n)}) \\ y = y_{\tilde{T}} - h_G(\xi, v, \dots, v^{(n)}) \end{cases}$$

initialized at $(x_0, \phi(x_0), z_0)$, is identically zero. This, together with (11.18), implies

$$\begin{aligned}
\frac{\partial y^{(k)}(w, x_0, \phi(x_0), z_0)}{\partial w_i} &= \frac{\partial y^{(k)}_{\tilde{T}}(w, x_0, z_0)}{\partial w_i} - \frac{\partial y^{(k)}_{G\tilde{H}}(w, x_0, \phi(x_0), z_0)}{\partial w_i} \\
&= \frac{\partial y^{(k)}_{\tilde{T}}(w, x_0, z_0)}{\partial w_i} - \frac{\partial y^{(k)}_{G}(y_{\tilde{H}}(w, z_0), \phi(x_0))}{\partial w_i} \\
&= \frac{\partial y^{(k)}_{\tilde{T}}(w, x_0, z_0)}{\partial w_i} \\
&\quad - \frac{\partial y^{(k)}_{G}(y_{\tilde{H}}(w, z_0), \phi(x_0))}{\partial y_{\tilde{H}}} \cdot \frac{\partial y_{\tilde{H}}(w, z_0)}{\partial w_i} \\
&= \frac{\partial y^{(k)}_{\tilde{T}}(w, x_0, z_0)}{\partial w_i} \\
&= 0, \qquad \text{for all } i \geq \rho + 1 \ \text{ and all } k
\end{aligned}$$

Therefore, $\rho \begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix}$ is not greater than $\rho(H)$ and, as a consequence,

$$\rho \begin{pmatrix} T \\ H \end{pmatrix} = \rho \begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix} = \rho(H)$$

The general case can always be reduced to the previous one. If (11.18) does not hold, one can pick $\rho(H)$ independent output components of H that can be decoupled with a regular dynamic state feedback [49, 38, 125]. Then, the extended system H_E verifies $\dim(\mathcal{G}_E \cup \mathcal{R}_{H_E}) = m - \rho(H_E)$. Since any solution of the GLFP concerning T and H also solves that concerning T_E and H_E and since the regular dynamic state feedback does not affect the system's rank, the conclusion follows from the first part. ∎

In general (11.17) is not sufficient for the solvability of the GLFP. However, under (11.18) and an additional technical condition, which assures the possibility of expressing z locally as a function of the output and its derivatives, it is possible to get a local result. More precisely, it is possible, for any z_0 in an open and dense subset of the state space, to find a neighborhood $\mathcal{D}_0$ and to show the existence of a compensator G and a map ϕ which achieve (11.16) for $z \in \mathcal{D}_0$. We will say, in this case, that the LFP is locally solvable.

Theorem 11.11. *The GLFP is locally solvable if the following conditions hold:*

i) $\rho \begin{pmatrix} T \\ H \end{pmatrix} = \rho(H)$;
ii) $\dim(\mathcal{G}_E \cup \mathcal{R}_{H_E}) = m - \rho(H)$;
iii) $\sum_{i \geq 0}(\rho(H) - s_i) = n$, *where* $n = \dim\, z$, $s_0 = 0$, *and the* s_i *are obtained by applying Singh's inversion algorithm to* H.

Proof. We consider a friend of $\mathcal{R}_H$, $u = \alpha(z) + \beta(z)w$, as in the proof of Theorem 11.10, and we use the notations introduced there. By the rank equality $\rho \begin{pmatrix} T \\ H \end{pmatrix} = \rho \begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix} = \rho(H)$, since (11.19) holds, the input components w_i, with $i \geq \rho + 1$, do not affect the output of $\begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix}$. By applying Singh's inversion algorithm to $\tilde{H}$,

$$\bar{w}_1 = \psi(z, v, \dot{v}, \ldots, v^{(n)}) \tag{11.20}$$

where ψ is a meromorphic function of its arguments and, in particular, it is defined for all z in an open dense subset of the state space. Moreover, using arguments as in ([87], §4), one can show that, by (iii), the Jacobian matrix

$$\frac{\partial}{\partial z} \begin{pmatrix} v - h_{\tilde{H}}(z) \\ \hat{v}_1 - \hat{v}_1(z, \tilde{v}_1) \\ \vdots \\ \hat{v}_N - \hat{v}_N(z, \tilde{v}_1, \ldots, \tilde{v}_1^{(N-1)}, \ldots, \tilde{v}_N)) \end{pmatrix}$$

whose elements are obtained by applying Singh's algorithm to $\tilde{H}$, has rank n. Then, for any z_0 in an open and dense subset of the state space, there exists a neighborhood $\mathcal{D}_0$ of z_0 such that $z = \chi(v, \dot{v}, \ldots, v^{(\nu)})$ for $z \in \mathcal{D}_0$. By substituting in (11.20), $\bar{w}_1 = \bar{\psi}(v, \dot{v}, \ldots, v^{(\nu)})$. Now, writing the state equation of $\begin{pmatrix} \tilde{T} \\ \tilde{H} \end{pmatrix}$ as

$$\begin{aligned} \dot{x} &= f_1(x, z) + g_1(x, z)\bar{w}_1 + g_2(x, z)\bar{w}_2 \\ \dot{z} &= f_2(z) + g_3(z)w \end{aligned}$$

we can consider the system

$$\begin{aligned} G &= \begin{cases} \dot{\xi} = f_1(\xi, \chi(v, \dot{v}, \ldots, v^{(\nu)})) + g_1(v, \dot{v}, \ldots, v^{(\nu)}))\bar{\psi}(v, \dot{v}, \ldots, v^{(\nu)})) \\ y_G = h(\xi) \end{cases} \\ &= \begin{cases} \dot{\xi} = f_G(\xi, v, \dot{v}, \ldots, v^{(\nu)}) \\ y_G = h(\xi) \end{cases} \end{aligned}$$

and we claim that (G, φ), where φ is the identity map, is a solution of the GLFP relative to $\mathcal{D}_0$. By inspection, one sees that the output trajectory $Y(w, x_0, \varphi(x_0), z_0)$ of the system

$$\begin{aligned} \dot{x} &= f_1(x, z) + g_1(x, z)\bar{w}_1 + g_2(x, z)\bar{w}_2 \\ \dot{z} &= f_2(z) + g_3(z)w \\ \dot{\xi} &= f_1(\xi, z) + g_1(\xi, z)\bar{w}_1 \\ y &= h(x) - h(\xi) \end{aligned}$$

is identically zero for all w. Inverting the feedback $u = \alpha(z) + \beta(z)$, we obtain $y_T(u, x_0) = y_{GH}(u, \varphi(x_0), z_0)$. ∎

Example 11.12. Let the systems

$$T = \begin{cases} \dot{x} = u \\ y_T = x \end{cases}$$

and

$$H = \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u \\ v = z_1^2 \end{cases}$$

be given. The conditions (i), (ii) of Theorem 11.10 are clearly verified as well as (iii) because $\rho(H) = 1$, $\dim z = 2$, $s_1 = 0$, $s_2 = 1$. In this case, there is no need to apply any feedback. By Singh's inversion algorithm, $u = \psi(z, v, \dot{v}, \ddot{v}) = (\ddot{v} - 2z_2^2)/2z_1$ and $v - z_1^2 = 0, \dot{v} - 2z_1 z_2 = 0$. Since

$$\frac{\partial}{\partial z} \begin{pmatrix} v - z_1^2 \\ \dot{v} - 2z_1 z_2 \end{pmatrix} = \begin{pmatrix} -2z_1 & 0 \\ z_2 & z_1 \end{pmatrix}$$

has rank 2 for $z_1 \neq 0$, we can express z as a function of $v, \dot{v}$ in the neighborhood of any point for which $z_1 \neq 0$. In particular, here,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \chi(v, \dot{v}) = \begin{pmatrix} \sqrt{v} \\ \dot{v}/2\sqrt{v} \end{pmatrix} \qquad \text{for } z_1 > 0$$

Then, the compensators

$$G_1 = \begin{cases} \dot{\xi} = \dfrac{\ddot{v} - \dot{v}^2/2v}{2\sqrt{v}} \\ y_G = \xi \end{cases}$$

and

$$G_2 = \begin{cases} \dot{\xi} = -\dfrac{\ddot{v} - \dot{v}^2/2v}{2\sqrt{v}} \\ y_G = \xi \end{cases}$$

together with the identity map, are local solutions of the GLFP, respectively, for $z_1 > 0$ and for $z_1 < 0$.

When a proper compensator is sought, the necessary condition (11.17) has to be strengthened into the equality of structures at infinity, and one obtains the following result.

Theorem 11.13. *The LFP is solvable with a proper compensator*

$$G = \begin{cases} \dot{\xi} = f_G(\xi, v) \\ y_G = h_G(\xi, v) \end{cases}$$

only if

$$\rho_i \begin{pmatrix} T \\ H \end{pmatrix} = \rho_i(H) \tag{11.21}$$

for all $i \geq 1$.

Proof. Let $\mathcal{K}$ denote the field of meromorphic functions in the variables $(x, z, u, \ldots, u^{(N-1)})$ where $N = \dim x + \dim z$. By definition,

$$\begin{aligned} \rho_i \begin{pmatrix} T \\ H \end{pmatrix} &= \dim \operatorname{span}_{\mathcal{K}} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\dot{y}_T, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_T^{(i)}, \mathrm{d}v^{(i)}\} \\ &\quad - \dim \operatorname{span}_{\mathcal{K}} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\dot{y}_T, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_T^{(i-1)}, \mathrm{d}v^{(i-1)}\} \end{aligned}$$

Denoting by $\mathcal{K}$' the field of meromorphic functions in the variables

$$(x, z, \xi, u, \ldots, u^{(N-1)}),$$

since neither T nor H depend on ξ,

$$\begin{aligned} \rho_i \begin{pmatrix} T \\ H \end{pmatrix} &= \dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{y}_T, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_T^{(i)}, \mathrm{d}v^{(i)}\} \\ &\quad - \dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{y}_T, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_T^{(i-1)}, \mathrm{d}v^{(i-1)}\} \end{aligned}$$

Since $y_{GH} = y_T$, one can substitute $\mathrm{d}y_{GH}^{(k)}$ in $\mathrm{d}y_T^{(k)}$ for all k in the equation above, thus

$$\rho_i \begin{pmatrix} T \\ H \end{pmatrix} = \dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{y}_{GH}, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_{GH}^{(i)}, \mathrm{d}v^{(i)}\}$$
$$-\dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{y}_{GH}, \mathrm{d}\dot{v}, \ldots, \mathrm{d}y_{GH}^{(i-1)}, \mathrm{d}v^{(i-1)}\}$$

Moreover, by the properness of (GH), we have also that

$$\mathrm{d}y_{GH}^{(k)} \in \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(k)}\}$$

thus

$$\rho_i \begin{pmatrix} T \\ H \end{pmatrix} = \dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(i)}\}$$
$$-\dim \operatorname{span}_{\mathcal{K}'} \{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\xi, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(i-1)}\}$$

Let $\mathcal{K}''$ denote the field of meromorphic functions in the variables $(z, u, \ldots, u^{(n-1)})$ where $n = \dim z$. Since

$$\mathrm{d}v^{(k)} \in \operatorname{span}_{\mathcal{K}''} \{\mathrm{d}z, \mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k-1)}\}$$

for all $k \leq n$, we get finally

$$\begin{aligned}\rho_i \begin{pmatrix} T \\ H \end{pmatrix} &= \dim \operatorname{span}_{\mathcal{K}''} \{\mathrm{d}z, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(i)}\} \\ &\quad -\dim \operatorname{span}_{\mathcal{K}''} \{\mathrm{d}z, \mathrm{d}\dot{v}, \ldots, \mathrm{d}v^{(i-1)}\} \\ &= \rho_i(H)\end{aligned}$$

∎

12

Measured Output Feedback Control Problems

For all control laws designed in the previous chapters, it was assumed that all state variables were available for measurement. This is seldom the case and two issues then exist:

- either the state variables are estimated by a (nonlinear) observer;
- a static or dynamic output feedback is sought directly to solve the control problem considered.

In this chapter, we will investigate the second option and solve some control problems either by output feedback or by measurement feedback.

12.1 Input-output Linearization

The second option was considered in [113, 114, 162]to solve the input-output linearization problem via static or dynamic measured output feedback.

12.1.1 Input-output Linearization via Static Output Feedback: the SISO Case

It is shown that linearization by input-output injection plays a crucial role in the solution of the feedback linearization problem by static output feedback.

Consider the nonlinear control system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{12.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and f, h are meromorphic functions of their arguments.

Find, if possible, a static output feedback $u = \alpha(y, v)$ such that the closed-loop has a linear input-output relation. Let $\bar{n} = \dim \mathcal{O}_\infty$. The solution of the problem is provided by

Theorem 12.1. *Assume that the relative degree of y is finite. The input-output linearization problem is solvable for (12.1) by static output feedback if and only if*

(i) $\mathrm{d}y^{(\bar{n})}$ *is linearizable by* $\bar{n}$ *output injections* $\phi_1(y,u), \ldots, \phi_{\bar{n}}(y,u)$*;*
(ii) $\dim_{I\!R}\left(\mathrm{span}_{I\!R}\{\mathrm{d}y, \mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_{\bar{n}}\}\right) = \dim_{\mathcal{K}}\left(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_{\bar{n}}\}\right)$.

Proof. Necessity: Let $u = \alpha(y,v)$ be the linearizing output feedback and $\gamma(y,u)$ its inverse, *i.e.*, $v = \gamma(y,u)$. Denote by r the relative degree of the output y. The linearized closed-loop system is described by the following equation:

$$\mathrm{d}y^{(\bar{n})} = \lambda_1 \mathrm{d}y^{(n-1)} + \cdots + \lambda_{\bar{n}} \mathrm{d}y + \beta_r \mathrm{d}v^{(\bar{n}-r)} + \cdots + \beta_{\bar{n}} \mathrm{d}v$$

where the λ_i's and β_i's are in $I\!R$. Substitute $v = \gamma(y,u)$ and get

$$\mathrm{d}y^{(\bar{n})} = \lambda_1 \mathrm{d}y^{(n-1)} + \cdots + \lambda_{\bar{n}} \mathrm{d}y + \beta_r \mathrm{d}[\gamma(y,u)]^{(\bar{n}-r)} + \cdots + \beta_{\bar{n}} \mathrm{d}[\gamma(y,u)].$$

Let $\mathrm{d}\phi_i(y,u) = \lambda_i \mathrm{d}y$ for $i = 1, \cdots, r-1$ and $\mathrm{d}\phi_i(y,u) = \lambda_i \mathrm{d}y + \beta_i \mathrm{d}\gamma(y,u)$ for $i = r, \cdots, \bar{n}$. Thus, the conditions (i) and (ii) in Theorem 12.1 are necessarily fulfilled.
Sufficiency: Conditions (i) and (ii) in Theorem 12.1 yield

$$\phi(y,u) = \lambda_i y + \beta_i \gamma(y,u)$$

for $i = 1, ..., \bar{n}$, and

$$\mathrm{d}y^{(\bar{n})} = \lambda_1 \mathrm{d}y^{(n-1)} + \cdots + \lambda_{\bar{n}} \mathrm{d}y + \beta_1 \mathrm{d}[\gamma(y,u)]^{(\bar{n}-1)} + \cdots + \beta_{\bar{n}} \mathrm{d}[\gamma(y,u)]$$

Since the relative degree is finite, the equation $\gamma(y,u) = v$ can be solved in u which yields the required output feedback. ∎

Example 12.2. Consider a simple inverted pendulum of length l as in Figure 12.1. Its model is standard and is given by

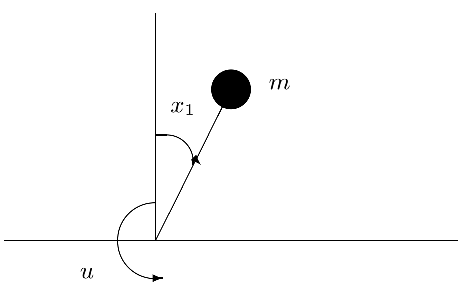

Fig. 12.1. Inverted pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 x_2 + k_2(\sin x_1) + k_3 u \\ y &= x_1 \end{aligned} \tag{12.2}$$

The angular position of the pendulum with respect to the vertical is denoted by x_1. The input torque is u, $k_1 = -f_c/ml^2$, $k_2 = g/l$, $k_3 = 1/ml^2$S, m is the point mass at the end of the pendulum, f_c is a viscous friction coefficient, and g denotes the gravitational constant. The system fulfils

$$\ddot{y} = k_1 x_2 + k_2(\sin x_1) + k_3 u$$

and thus

$$\ddot{y} = k_1 \dot{y} + \phi_2(y, u)$$

Obviously, the input-output relation is linearized by the following output feedback:

$$u = \frac{1}{k_3}(v - k_2 \sin y)$$

and the closed-loop system input-output relation is $\ddot{y} = k_1 \dot{y} + v$.

12.1.2 Input-output Linearization by Dynamic Output Feedback

In practice, the conditions of Theorem 12.1 are strong; it is seldom that there exists a static output feedback solution of the problem considered. Thus, it is interesting to search for a solution in a much wider class of compensators that still can be implemented that easily: the class of dynamic output feedbacks.

Consider first the following introductory example:

$$\begin{aligned} \dot{x}_1 &= x_2 - \frac{1}{2}\ln(x_1 + 2u) \\ \dot{x}_2 &= \frac{x_1 + u}{x_1 + 2u} + \frac{1}{2}\left(\frac{x_2 - \frac{1}{2}\ln(x_1 + 2u)}{x_1 + 2u}\right) \\ y &= x_1 \end{aligned} \tag{12.3}$$

whose input-output differential equation is

$$\ddot{y} = \phi_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi_1(y, u)\right], y, u\right) \tag{12.4}$$

where $\phi_1(y, u) = u$, $\phi_2(w, y, u) = \frac{y+u-w}{y+2u}$. Although this structure is different from the one that characterizes linearity up to output injections, it allows the design of a dynamic compensator that solves the input-output linearization: set

$$\begin{aligned} \eta &= u \\ v &= \tfrac{y+u-\dot{\eta}}{y+2u} \end{aligned} \tag{12.5}$$

Then, the dynamic output feedback is deduced

$$\begin{aligned} \dot{\eta} &= y + \eta - (y + 2\eta)v \\ u &= \eta \end{aligned} \tag{12.6}$$

and the closed-loop system is linear since its input-output equation is $\ddot{y} = v$.

What was operated on this example can be done on general nonlinear systems. Let us first state the problem before giving a sufficient condition under which a solution exists and can be constructed.

Problem statement : Given system (12.1), find if possible, a dynamic output feedback

$$\begin{aligned} u &= H(y, \eta, v) \\ \dot{\eta} &= F(y, \eta, v) \end{aligned} \tag{12.7}$$

such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, H(h(x), \eta, v)) \\ \dot{\eta} &= F(h(x), \eta, v) \\ y &= h(x) \end{aligned} \tag{12.8}$$

is diffeomorphic to

$$\begin{aligned} \dot{\zeta}^1 &= A\zeta^1 + bv \\ \dot{\zeta}^2 &= \bar{f}^2(\zeta, \eta, v) \\ y &= c\zeta^1 \end{aligned} \tag{12.9}$$

where, $\eta \in I\!R^q$, $\zeta^1 \in I\!R^{\bar{n}}, \zeta^2 \in I\!R^{n+q-\bar{n}}$, and (c, A) is an observable pair. Two sufficient conditions are given successively below. Both generalize the structure (12.4) in the introductory example.

Theorem 12.3. *The system (12.1) can be linearized by dynamic output feedback if*

$$\mathrm{d}y^{(\bar{n})} = \lambda_1 \mathrm{d}y^{(\bar{n}-1)} + \cdots + \lambda_{r-1} \mathrm{d}y^{(\bar{n}-r+1)} + \mathrm{d}\Phi \tag{12.10}$$

where

$$\Phi = \phi_{\bar{n}}(\cdot, y, u) \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\bar{n}-1}(\cdot, y, u) \circ \ldots \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_{r+1}(\cdot, y, u) \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_r(y, u)$$

and $\lambda_i \in I\!R \quad (i = 1, \ldots, r-1)$.

Proof. Consider the system (12.1).
Let

$$\begin{aligned} \eta_1 &:= \phi_r(y, u) \\ \eta_2 &:= \phi_{r+1}(\dot{\eta}_1, y, u) \\ \eta_3 &:= \phi_{r+2}(\dot{\eta}_2, y, u) \\ &\vdots \\ \eta_{\bar{n}-r} &:= \phi_{\bar{n}-1}(\dot{\eta}_{\bar{n}-r-1}, y, u) \\ v &:= \phi_{\bar{n}}(\dot{\eta}_{\bar{n}-r}, y, u) \end{aligned} \tag{12.11}$$

From the definition of the relative degree r, $\dfrac{\partial \phi_r}{\partial u} \not\equiv 0$ and $\dfrac{\partial \phi_i}{\partial \dot{\phi}_{i-1}} \not\equiv 0$ for $i = r+1, \ldots, \bar{n}$. Thus, the following dynamic compensator is well defined:

$$\begin{aligned} u &= \phi_r^{-1}(y, \eta_1) \\ \dot{\eta}_i &= \phi_{r+i}^{-1}(\eta_{i+1}, y, \phi_r^{-1}(y, \eta_1)) \quad i = 1, \ldots, \bar{n} - r - 1 \\ \dot{\eta}_{\bar{n}-r} &= \phi_{\bar{n}}^{-1}(v, y, \phi_r^{-1}(y, \eta_1)) \end{aligned} \tag{12.12}$$

Substitute (12.12) in (12.10) and get the closed-loop system behavior:

$$\mathrm{d}y^{(\bar{n})} = \lambda_1 \mathrm{d}y^{(\bar{n}-1)} + \cdots + \lambda_{r-1} \mathrm{d}y^{(\bar{n}-r+1)} + \mathrm{d}v \tag{12.13}$$

System (12.1) has been linearized by dynamic output feedback. ∎

A way to check condition (12.10) is provided by the following algorithm. It allows us to compute the required functions ϕ_i $(i = 1, \ldots, \bar{n})$ whenever they exist. Denote $\mathcal{F}^k = \mathcal{E}\{\mathrm{d}y, ..., \mathrm{d}y^{(k-1)}, \mathrm{d}u, ..., \mathrm{d}u^{(k-1)}\}$.

Algorithm 12.4 *Assume that,*

$$\dim \mathcal{F}^{\bar{n}} = 2\bar{n}. \tag{12.14}$$

Initial check: $\mathrm{d}y^{(\bar{n})} \in E^{\bar{n}}$. *If false, stop! If true, denote* $\omega_1 = \mathrm{d}y^{(\bar{n})}$.

Step i (i=1,...,r-1).
Pick functions ξ_i *such that* $\omega_i - \xi_i \mathrm{d}y^{(\bar{n}-i)} \in E^{\bar{n}-i}$. *Let* $\bar{\omega}_i = \xi_i \mathrm{d}y$. *Check:* $\bar{\omega}_i \in \mathrm{span}_{I\!R}\{\mathrm{d}y\}$. *If false, stop! If true, define* $\omega_{i+1} = \omega_i - \bar{\omega}_i$.

Step r.
Pick functions $\xi_r, \theta_r \in \mathcal{K}$ *such that* $\omega_r - \xi_r \mathrm{d}y^{(\bar{n}-r)} - \theta_r \mathrm{d}u^{(\bar{n}-r)} \in E^{\bar{n}-r}$. *Let* $\bar{\omega}_r = \xi_r \mathrm{d}y + \theta_r \mathrm{d}u$. *Check:* $\mathrm{d}\bar{\omega}_r \wedge \bar{\omega}_r = 0$. *If false, stop! If true, define* $\phi_r(y, u)$ *such that*

$$\bar{\omega}_r = \lambda_r \mathrm{d}\phi_r \tag{12.15}$$

where $\lambda_r \in \mathcal{K}$. *Denote* $z_{r+1} = \dot{\phi}_r(y, u)$. *If* $\dfrac{\partial \phi_r}{\partial y} \neq 0$, *rewrite* ω_r *so that* $\omega_r \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_{r+1}^{(\bar{n}-r-1)}, \ldots, \mathrm{d}z_2, \mathrm{d}u^{(\bar{n}-r-1)}, \ldots, \mathrm{d}u, \mathrm{d}y\}$.
If $\dfrac{\partial \phi_r}{\partial y} = 0$ *et* $\dfrac{\partial \phi_r}{\partial u} \neq 0$, *rewrite* ω_r *so that*

$$\omega_r \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_{r+1}^{(\bar{n}-r-1)}, \ldots, \mathrm{d}z_{r+1}, \mathrm{d}y^{(\bar{n}-r-1)}, \ldots, \mathrm{d}y, \mathrm{d}u\}$$

Step ℓ (ℓ=r+1, ..., $\bar{n}$-1).
Pick $\xi_\ell, \theta_\ell, \mu_\ell \in \mathcal{K}$ *such that*

$$\omega_r - \mu_\ell \mathrm{d}z_\ell^{(\bar{n}-\ell)} - \xi_\ell \mathrm{d}y^{(\bar{n}-\ell)} - \theta_\ell \mathrm{d}u^{(\bar{n}-\ell)} \in E^{\bar{n}-\ell}$$

Set $\bar{\omega}_\ell = \mu_\ell \mathrm{d}z_\ell + \xi_\ell \mathrm{d}y + \theta_\ell \mathrm{d}u$. *Check:* $\mathrm{d}\bar{\omega}_\ell \wedge \bar{\omega}_\ell \wedge \mathrm{d}\phi_{\ell-1} = 0$. *If false, stop! If true, define* $\phi_\ell(z_\ell, y, u)$ *such that*

$$\bar{\omega}_\ell = \lambda_\ell \mathrm{d}\phi_\ell + \gamma_\ell \mathrm{d}\phi_{\ell-1} \tag{12.16}$$

where $\gamma_\ell, \lambda_\ell \in \mathcal{K}$. *Denote* $z_{\ell+1} = \dot{\phi}_\ell(z_\ell, y, u)$. *Rewrite* ω_r *so that*

$$\omega_r \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_{\ell+1}^{(\bar{n}-\ell-1)}, \ldots, \mathrm{d}z_{\ell+1}, \mathrm{d}y^{(\bar{n}-\ell-1)}, \ldots, \mathrm{d}y, \mathrm{d}u^{(\bar{n}-\ell-1)}, \ldots, \mathrm{d}u\}$$

Step $\bar{n}$.
By definition, $\omega_r \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_{\bar{n}}, \mathrm{d}y, \mathrm{d}u\}$, *and*

$$\mathrm{d}\omega_r = 0$$

End of the Algorithm.

Theorem 12.5. *Under assumption (12.14),* $\mathrm{d}y^{(\bar{n})} \in E$ *can be rewritten in the form (12.10) if and only if* $\mathrm{d}y^{(\bar{n})} \in E^{\bar{n}}$ *and*

•

$$\mathrm{d}\bar{\omega}_r \wedge \bar{\omega}_r = 0 \tag{12.17}$$

$$\mathrm{d}\bar{\omega}_\ell \wedge \bar{\omega}_\ell \wedge \mathrm{d}\phi_{\ell-1} = 0, \quad \ell = r+1, \ldots, \bar{n}-1 \tag{12.18}$$

• *there exist* $\xi_1, \ldots, \xi_{r-1} \in I\!R$, *such that*

$$\mathrm{d}y^{(\bar{n})} - \sum_{i=1}^{r-1} \xi_i \mathrm{d}y^{(\bar{n}-i)} \in E^{\bar{n}-r} \tag{12.19}$$

Proof. Necessity: Thanks to Theorem 12.3, the necessity of (12.19) is obvious. Assume that there exist $\bar{n}-r+1$ functions ϕ_i $(i = r, \ldots, \bar{n})$ so that $\mathrm{d}y^{(\bar{n})}$ reads as in condition (i) of Theorem 12.3. Then, from (12.15) and (12.16), the differential 1-forms $\bar{\omega}_i$ can be rewritten as follows:

$$\begin{aligned}
\bar{\omega}_r &= \frac{\partial \phi_{\bar{n}}}{\partial \dot{\phi}_{\bar{n}-1}} \frac{\partial \phi_{\bar{n}-1}}{\partial \dot{\phi}_{\bar{n}-2}} \cdots \frac{\partial \phi_{r+1}}{\partial \dot{\phi}_r} \mathrm{d}\phi_r(y, u) \\
\bar{\omega}_{r+1} &= \frac{\partial \phi_{\bar{n}}}{\partial \dot{\phi}_{\bar{n}-1}} \frac{\partial \phi_{\bar{n}-1}}{\partial \dot{\phi}_{\bar{n}-2}} \cdots \frac{\partial \phi_{r+2}}{\partial \dot{\phi}_{r+1}} \mathrm{d}\phi_{r+1}(\dot{\phi}_r, y, u) + \gamma_{r+1} \mathrm{d}\phi_r \\
&\vdots \\
\bar{\omega}_{\bar{n}-1} &= \frac{\partial \phi_{\bar{n}}}{\partial \dot{\phi}_{\bar{n}-1}} \mathrm{d}\phi_{\bar{n}-1}(\dot{\phi}_{\bar{n}-2}, y, u) + \gamma_{\bar{n}-1} \mathrm{d}\phi_{\bar{n}-2}
\end{aligned}$$

This proves the necessity of (12.17) and of (12.18).
Sufficiency: The sufficiency of Theorem 12.5 follows from Algorithm 12.4. ∎

A weaker, but more complex, sufficient condition is given now.

Theorem 12.6. *System (12.1) can be linearized by dynamic output feedback if*

$$
\begin{aligned}
\exists\ \tilde{n},\ q \in I\!N \quad \mathrm{d}y^{(\tilde{n})} = {} & \lambda_1 \mathrm{d}y^{(\tilde{n}-1)} + \cdots + \lambda_{\tilde{n}} \mathrm{d}y \\
& + \mathrm{d}\Big[\phi_q(\ \cdot\ , \dot{\phi}_{q-2}, \ldots, \dot{\phi}_1, y, u) \\
& \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_{q-1}(\ \cdot\ , \dot{\phi}_{q-3}, \ldots, \dot{\phi}_1, y, u) \circ \ldots \\
& \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(\ \cdot\ , y, u) \circ \frac{\mathrm{d}}{\mathrm{d}t}\phi_1(y, u)\Big]
\end{aligned}
\tag{12.20}
$$

where $\frac{\partial \phi_1}{\partial u} \neq 0$*, and* $\lambda_i \in I\!R \quad (i = 1, \ldots, \tilde{n})$*.*

Remark 12.7. Since the observability index cannot decrease, $\tilde{n} \geq \bar{n}$, and since the relative degree cannot decrease either, $\tilde{n} - q + 1 \geq r$.

Proof (Proof of Theorem 12.6.). Consider system (12.1). Let

$$
\begin{aligned}
\eta_1 &:= \phi_1(y, u) \\
\eta_2 &:= \phi_2(\dot{\eta}_1, y, u) \\
\eta_3 &:= \phi_3(\dot{\eta}_2, \dot{\eta}_1, y, u) \\
&\ \vdots \\
\eta_{q-1} &:= \phi_{q-1}(\dot{\eta}_{q-2}, \ldots, \dot{\eta}_1, y, u) \\
v &:= \phi_q(\dot{\eta}_{q-1}, \ldots, \dot{\eta}_1, y, u)
\end{aligned}
\tag{12.21}
$$

From the definition of the relative degree, $\dfrac{\partial \phi_1}{\partial u} \not\equiv 0$ and $\dfrac{\partial \phi_i}{\partial \dot{\phi}_{i-1}} \not\equiv 0$ for $i = 1, \ldots, q$. Define the following dynamic output feedback:

$$
\begin{aligned}
\dot{\eta}_i &= \phi_{i+1}^{-1}(\eta_{i+1}, \phi_i^{-1}, \ldots, \phi_2^{-1}, y, \phi_1^{-1}(y, \eta_1)) \quad i = 1, \ldots, q-2 \\
\dot{\eta}_{q-1} &= \phi_q^{-1}(v, \phi_{q-1}^{-1}, \ldots, \phi_2^{-1}, y, \phi_1^{-1}(y, \eta_1)) \\
u &= \phi_1^{-1}(y, \eta_1)
\end{aligned}
\tag{12.22}
$$

Substitute (12.22) into (12.20) and get the input-output relation for the closed-loop system:

$$
\mathrm{d}y^{(\tilde{n})} = \lambda_1 \mathrm{d}y^{(\tilde{n}-1)} + \cdots + \lambda_{\tilde{n}} \mathrm{d}y + \mathrm{d}v \tag{12.23}
$$

Thus, system (12.1) has been linearized. ∎

If $q \leq \bar{n}$, an algorithmic necessary and sufficient condition is given by Algorithm 12.9.

The following lemma is required to introduce Algorithm 12.9 ([81], Lemma 2.6).

Lemma 12.8. *Consider a cospace $\Omega \subset \Omega'$ and a 1-form $\omega \notin \Omega$ and $\omega \in \Omega'$. Then, there exists $\pi \in \Omega$ such that $\mathrm{d}(\omega + \pi) \wedge (\omega + \pi) = 0$ if and only if*

$$\dim(\mathrm{span}\{\omega\} + \Omega)^* \geq \dim(\Omega)^* + 1 \tag{12.24}$$

where Ω^* denotes the largest integrable subspace contained in a given space Ω.

Algorithm 12.9 *Assume that $q \leq \bar{n}$.*
Initial check: $\exists \alpha_1, \ldots, \alpha_{\tilde{n}-q} \in I\!R$ such that $\mathrm{d}y^{(\tilde{n})} - \sum_{i=1}^{\tilde{n}-q} \alpha_i \mathrm{d}y^{(\tilde{n}-i)} \in E^q$. If false, stop!
If true, denote $\omega = \mathrm{d}y^{(\tilde{n}} - \sum_{i=1}^{\tilde{n}-q} \alpha_i \mathrm{d}y^{(\tilde{n}-i)}$.

Step 1.
Check 1.1: $\omega \in E^q$. If false, stop!
If true, pick functions $\xi_1, \theta_1 \in \mathcal{K}$ such that $\omega - \xi_1 \mathrm{d}y^{(q-1)} - \theta_1 \mathrm{d}u^{(q-1)} \in E^{q-1}$.
Define a differential 1-form $\bar{\omega}_1$ such that $\bar{\omega}_1 = \xi_1 \mathrm{d}y + \theta_1 \mathrm{d}u$.
Check 1.2: $d\bar{\omega}_1 \wedge \bar{\omega}_1 = 0$. If false, stop!
If true, let $\phi_1(y,u)$ such that $\bar{\omega}_1 = \lambda_1 \mathrm{d}\phi_1$ where $\lambda_1 \in \mathcal{K}$. Let $z_1 = \dot{\phi}_1(y,u)$.

Step 2.
Check 2.1: $\omega \in E^{q-1} + \mathrm{span}\{\mathrm{d}z_1^{(q-2)}\}$. If false, stop!
If true, pick functions $\xi_2, \theta_2, \mu_{21} \in \mathcal{K}$ such that

$$\omega - \mu_{21} \mathrm{d}z_1^{(q-2)} - \xi_2 \mathrm{d}y^{(q-2)} - \theta_2 u^{(q-2)} \in E^{q-2}$$

Define a differential 1-form $\bar{\omega}_2$ such that $\bar{\omega}_2 = \mu_{21} \mathrm{d}z_1 + \xi_2 \mathrm{d}y + \theta_2 \mathrm{d}u$.
Define $\Omega_2 = \mathrm{span}\{\mathrm{d}\phi_1\}$.
Check 2.2: $\mathrm{d}\bar{\omega}_2 \wedge \bar{\omega}_2 \wedge \mathrm{d}\phi_1 = 0$. If false, stop!
If true, then there exist $\pi_2 \in \Omega_1$, $\lambda_2 \in \mathcal{K}$, and $\phi_2(z_1, y, u)$ such that $\bar{\omega_2} + \pi_2 = \lambda_2 \mathrm{d}\phi_2$. Let $z_2 = \dot{\phi}_2(z_1, y, u)$.

Step ℓ ($\ell \leq$ q-1).
Check l.1: $\omega \in E^{q-l+1} + \mathrm{span}\{\mathrm{d}z_1^{(q-l)}, \ldots, \mathrm{d}z_{l-1}^{(q-l)}\}$. If false, stop!
If true, pick functions $\xi_l \theta_l, \mu_{l,1}, \ldots, \mu_{l,l-1} \in K$ such that

$$\omega - \mu_{l,1} \mathrm{d}z_1^{(q-l)} - \ldots - \mu_{l,l-1} \mathrm{d}z_{l-1}^{(q-l)} - \xi_l \mathrm{d}y^{(q-l)} - \theta_l \mathrm{d}u^{(q-l)} \in E^{q-l}$$

Define a differential 1-form $\bar{\omega}_l$ such that

$$\bar{\omega}_l = \mu_{l,l-1} \mathrm{d}z_{l-1} + \cdots + \mu_{l,1} \mathrm{d}z_1 + \xi_l \mathrm{d}y + \theta_l \mathrm{d}u$$

Define the cospace

$$\Omega_l = \mathrm{span}\{\sum_{i=1}^{k-1} \frac{\partial \phi_k}{\partial \dot{z}_i} \mathrm{d}z_i, k = 1, \ldots, \min(l-1, q-l); \mathrm{d}\phi_1\}$$

Check l.2: $\dim(\mathrm{span}\{\bar{\omega}_l\} + \Omega_l)^* \geq \dim\Omega_l^* + 1$. *If false, stop!*
If true, then there exist $\pi_l \in \Omega_l$, $\lambda_l \in \mathcal{K}$ *and* $\phi_l(z_1, \ldots, z_{l-1}, y, u)$ *such that* $\bar{\omega}_l + \pi_l = \lambda_l \mathrm{d}\dot{\phi}_l$.
Denote $z_l = \dot{\phi}_l(z_{l-1}, \ldots, z_1 y, u)$.

Step q.
Check q.1: $\omega \in E^1 + \mathrm{span}\{\mathrm{d}z_1, \ldots, \mathrm{d}z_{q-1}\}$. *If false, stop!*
If true, pick functions $\xi_q, \theta_q, \mu_{q,1}, \ldots, \mu_{q,q-1} \in \mathcal{K}$ *such that*

$$\omega = \mu_{q,q-1}\mathrm{d}z_{q-1} + \cdots + \mu_{q,1}\mathrm{d}z_1 + \xi_q \mathrm{d}y + \theta_q \mathrm{d}u$$

Check q.2: $\mathrm{d}\omega \wedge \omega = 0$. *If false, stop!*
End of the Algorithm.

One defines a triangular endogenous dynamic output feedback as follows

$$\begin{aligned}
u &= F_1(y, \eta_1) \\
\dot{\eta}_1 &= F_2(y, \eta_1, \eta_2) \\
\dot{\eta}_2 &= F_3(y, \eta_1, \ldots, \eta_3) \\
&\vdots \\
\dot{\eta}_{q-2} &= F_{q-1}(y, \eta_1, \ldots, \eta_{q-1}) \\
\dot{\eta}_{q-1} &= F_q(y, \eta_1, \ldots, \eta_{q-1}, v)
\end{aligned} \tag{12.25}$$

Endogenous feedbacks have first been considered in [115]. Note that (12.12), (12.22) are triangular endogenous dynamic output feedbacks. This class of feedbacks, (12.20), is necessary for the dynamic output feedback linearization problem.

Theorem 12.10. *There exists a triangular endogenous dynamic output feedback that linearizes the input-output relation of system (12.1) if and only if condition (12.20) is fulfilled.*

Proof. Sufficiency: The sufficiency follows from the proof of Theorem 12.6 since the compensator described in (12.22) is a triangular endogenous dynamic output feedback.
Necessity: There exist (12.25) such that the linear closed-loop system reads

$$\mathrm{d}y^{(\tilde{n})} = \lambda_1 \mathrm{d}y^{(\tilde{n}-1)} + \cdots + \lambda_{\tilde{n}}\mathrm{d}y + \mu_1 \mathrm{d}v^{(s)} + \ldots + \mu_{s+1}\mathrm{d}v \tag{12.26}$$

for some integer s. Define the following triangular endogenous dynamic output feedback:

$$\begin{aligned}
v &= \xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_s &= \frac{1}{\mu_1}\left[w - \mu_2 \xi_{s-1} - \ldots - \mu_{s+1}\xi_1\right]
\end{aligned} \tag{12.27}$$

The composition of (12.25) with (12.27) maintains the structure of the triangular endogenous dynamic output feedback and yields

$$\mathrm{d}y^{(\tilde{n})} = \lambda_1 \mathrm{d}y^{(\tilde{n}-1)} + \cdots + \lambda_{\tilde{n}} \mathrm{d}y + \mathrm{d}w \tag{12.28}$$

Thus, without loss of generality, assume that the triangular endogenous dynamic output feedback (12.25) yields (12.28). From (12.25),

$$\begin{aligned} \eta_1 &= F_1^{-1}(y, u) \\ \eta_2 &= F_2^{-1}(\dot{\eta}_1, y, u) \\ \eta_3 &= F_3^{-1}(\dot{\eta}_2, \dot{\eta}_1, y, u) \\ &\vdots \\ w &= F_q^{-1}(\dot{\eta}_{q-1}, \ldots, \dot{\eta}_1, y, u) \end{aligned} \tag{12.29}$$

Thus, one obtains the following expression for w :

$$\begin{aligned} w = \Bigg[&\phi_q(\ \cdot\ , \dot{\phi}_{q-2}, \ldots, \dot{\phi}_1, y, u) \circ \frac{\mathrm{d}}{\mathrm{d}t} \phi_{q-1}(\ \cdot\ , \dot{\phi}_{q-3}, \ldots, \dot{\phi}_1, y, u) \circ \ldots \\ &\circ \frac{\mathrm{d}}{\mathrm{d}t} \phi_2(\ \cdot\ , \dot{\phi}_1, y, u) \circ \frac{\mathrm{d}}{\mathrm{d}t} \phi_1(y, u) \Bigg] \end{aligned} \tag{12.30}$$

where $\phi_i = F_i^{-1}$ for $i = 1, \ldots, q$. Combining (12.26) and (12.30), the result follows. ∎

Theorem 12.11. *Assume that* $\dim E^{\bar{n}} = 2\bar{n}$ *and* $q \leq \bar{n}$. *For* $\tilde{n} \geq \bar{n}$, $\mathrm{d}y^{(\tilde{n})}$ *can be written as (12.20) if and only if all checks in Algorithm 12.9 are fulfilled.*

Proof. The sufficiency obviously follows from the construction of Algorithm 12.9, since $\mathrm{d}\omega \wedge \omega = 0$ at the end of Step q.

The necessity is proven by induction. The necessity of the initial check as well as the necessity of the two checks in Step 1 are obvious. Assume that all checks in Steps $1, \ldots, l$, $l \leq q - 2$ are fulfilled; then, one shows that the two checks in Step $l + 1$ are fulfilled as well.

Let

$$\begin{aligned} z_1 &= \dot{\phi}_1(y, u) \\ &\vdots \\ z_l &= \dot{\phi}_l(z_{l-1}, \ldots, z_1, y, u) \end{aligned}$$

Then, by assumption, for some $\xi \in \mathcal{K}$,

$$\begin{aligned} \omega &= \xi \mathrm{d}\phi_q(\dot{\phi}_{q-1}, \ldots, \dot{\phi}_{l+1}, z_l, \ldots, z_1, y, u) \\ &= \xi \cdot \left(\frac{\partial \phi_n}{\partial \dot{\phi}_{n-1}} \mathrm{d}\dot{\phi}_{n-1} + \ldots + \frac{\partial \phi_q}{\partial \dot{\phi}_{l+1}} \mathrm{d}\dot{\phi}_{l+1} \right. \\ &\qquad \left. + \frac{\partial \phi_q}{\partial z_l} \mathrm{d}z_l + \ldots + \frac{\partial \phi_q}{\partial z_1} \mathrm{d}z_1 \right) \end{aligned} \tag{12.31}$$

Since $\mathrm{d}\phi_{l+1} \subset \mathrm{span}\{\mathrm{d}z_l, \dots, \mathrm{d}z_1, \mathrm{d}y, \mathrm{d}u\}$,

$$\mathrm{d}\dot{\phi}_{l+1} \subset \mathrm{span}\{\mathrm{d}\dot{z}_l \dots, \mathrm{d}\dot{z}_1, \mathrm{d}\dot{y}, \mathrm{d}\dot{u}, \mathrm{d}z_l, \dots, \mathrm{d}z_1, \mathrm{d}y, \mathrm{d}u\} \tag{12.32}$$

and

$$\begin{aligned} \mathrm{d}\phi_{l+2} &\subset \mathrm{span}\{\mathrm{d}\dot{\phi}_{l+1}, \dots, \mathrm{d}\dot{z}_1, \mathrm{d}y, \mathrm{d}u\} \\ &\subset \mathrm{span}\{\frac{\mathrm{d}(\mathrm{d}\phi_{l+1})}{\mathrm{d}t}, \mathrm{d}\dot{z}_l \dots, \mathrm{d}\dot{z}_1, \mathrm{d}y, \mathrm{d}u\} \\ &\overset{(12.32)}{\subset} \mathrm{span}\{\mathrm{d}\dot{z}_l \dots, \mathrm{d}\dot{z}_1, \mathrm{d}\dot{y}, \mathrm{d}\dot{u}, \mathrm{d}z_l, \dots, \mathrm{d}z_1, \mathrm{d}y, \mathrm{d}u\} \end{aligned} \tag{12.33}$$

Thus,

$$\mathrm{d}\dot{\phi}_{l+2} \subset \mathrm{span}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y}, \mathrm{d}u, \mathrm{d}\dot{u}, \mathrm{d}\ddot{u}, \mathrm{d}z_i, \mathrm{d}\dot{z}_i, \mathrm{d}\ddot{z}_i, i = 1, \dots, l\} \tag{12.34}$$

In a similar vein, one proves that

$$\begin{aligned} \mathrm{d}\phi_{q-1} \subset \mathrm{span}\{&\mathrm{d}y, \dots, \mathrm{d}y^{(q-l-2)}, \mathrm{d}u, \dots, \mathrm{d}u^{(q-l-2)}, \\ &\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(q-l-2)}, i = 1, \dots, l\} \end{aligned} \tag{12.35}$$

and

$$\begin{aligned} \mathrm{d}\dot{\phi}_{q-1} \subset \mathrm{span}\{&\mathrm{d}y, \dots, \mathrm{d}y^{(q-l-1)}, \mathrm{d}u, \dots, \mathrm{d}u^{(q-l-1)}, \\ &\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(q-l-1)}, i = 1, \dots, l\} \end{aligned} \tag{12.36}$$

Thus, from (12.31), (12.32), (12.34), and (12.36),

$$\omega \in E^{q-l} + \mathrm{span}\{\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(q-l-1)}\} \tag{12.37}$$

It is shown by induction that

$$\mathrm{span}\{\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(k)}\} \subset E^{i+k} \tag{12.38}$$

$$\mathrm{span}\{\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(q-l-2)}\} \subset E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_j^{(q-l-1)}, j = 1, \dots, i-1\} \tag{12.39}$$

for $i = 1, \dots, l$.

For instance, let us prove (12.39).

Equation (12.39) is fulfilled for $i = 1$ due to the definition of z_1. Assume that it is fulfilled for $i - 1$. Since

$$\begin{aligned} \mathrm{span}\{\mathrm{d}z_i, \dots, \mathrm{d}z_i^{(q-l-2)}\} &\subset E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_j, \dots, \\ &\qquad \mathrm{d}z_j^{(q-l-1)}, j = 1, \dots, i-1\} \\ &\subset E^{q-l-1} + \\ &\qquad + \mathrm{span}\{\mathrm{d}z_j^{(q-l-1)}, j = 1, \dots, i-1\} \end{aligned} \tag{12.40}$$

where the second inclusion yields from the induction hypothesis, (12.39) is also fulfilled for i.

Combining (12.37) and (12.40),

$$\omega \in E^{q-l} + \mathrm{span}\{\mathrm{d}z_j^{(q-l-1)}, j = 1, \ldots, i-1\}$$

It is thus necessary to fulfill the first check of Step $l+1$.

To prove the necessity of the second check of Step $l+1$, let us develop (12.31) which is obtained from the definition of z_i:

$$\begin{aligned}
&\omega - \xi \cdot \left[\frac{\partial \phi_q}{\partial \dot{\phi}_{q-1}} \cdots \frac{\partial \phi_{l+2}}{\partial \dot{\phi}_{l+1}}\right] \left(\sum_{i=1}^{l} \frac{\partial \phi_{l+1}}{\partial z_i} \mathrm{d}z_i^{(q-l-1)}\right. \\
&\left. + \frac{\partial \phi_{l+1}}{\partial y} \mathrm{d}y^{(q-l-1)} + \frac{\partial \phi_{l+1}}{\partial u} \mathrm{d}u^{(q-l-1)}\right) \\
&\in E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_i^j, i = 1, \ldots, l; j = 0, \ldots, q-l-2\} \\
&\overset{(12.38)}{=} E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_i^j, \\
&i = 1, \ldots, \min(l, q-l+1); j = n-l-i-1, \ldots, n-l-2\}
\end{aligned} \tag{12.41}$$

One has,

$$\mathrm{d}z_1^{(q-l-2)} = \frac{\partial \phi_1}{\partial y} \mathrm{d}y^{(q-l-1)} + \frac{\partial \phi_1}{\partial u} \mathrm{d}u^{(q-l-1)} \left(\mathrm{mod}\ E^{q-l-1}\right) \tag{12.42}$$

$$\begin{aligned}
\mathrm{d}z_2^{(q-l-3)} &= \frac{\partial \phi_2}{\partial z_1} \mathrm{d}z_1^{(q-l-2)} \left(\mathrm{mod}\ E^{q-l-2}\right. \\
&\quad \left. + \mathrm{span}\{\mathrm{d}z_i^{(j)}, j = 0, \ldots, q-l-3\}\right) \\
&\overset{(12.38)}{=} \frac{\partial \phi_2}{\partial z_1} \mathrm{d}z_1^{(q-l-2)} \left(\mathrm{mod}\ E^{q-l-1}\right) \\
&\overset{(12.42)}{=} \frac{\partial \phi_2}{\partial z_1} \left(\frac{\partial \phi_1}{\partial y} \mathrm{d}y^{(q-l-1)}\right. \\
&\quad \left. + \frac{\partial \phi_1}{\partial u} \mathrm{d}u^{(q-l-1)}\right) \left(\mathrm{mod}\ E^{q-l-1}\right)
\end{aligned} \tag{12.43}$$

$$\begin{aligned}
\mathrm{d}z_2^{(q-l-2)} &= \frac{\partial \phi_2}{\partial z_1} \mathrm{d}z_1^{(q-l-1)} \\
&\quad \left(\mathrm{mod}\ E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_1^{(j)}, j = 0, \ldots, q-l-2\}\right) \\
&\overset{(12.38)}{=} \frac{\partial \phi_2}{\partial z_1} \mathrm{d}z_1^{(q-l-1)} \left(\mathrm{mod}\ E^{q-l-1} + \mathrm{span}\{\mathrm{d}z_1^{(q-l-2)}\}\right) \\
&\overset{\exists \mu_2 \in \mathcal{K}, (12.42)}{=} \frac{\partial \phi_2}{\partial z_1} \mathrm{d}z_1^{(q-l-1)} + \mu_2 \left(\frac{\partial \phi_1}{\partial y} \mathrm{d}y^{(q-l-1)}\right. \\
&\quad \left. + \frac{\partial \phi_1}{\partial u} \mathrm{d}u^{(q-l-1)}\right) \left(\mathrm{mod}\ E^{q-l-1}\right)
\end{aligned} \tag{12.44}$$

and more generally, one shows by induction that there exists $\mu_{kij} \in \mathcal{K}$ such that

$$\mathrm{d}z_k^{(q-l-k-1)} = \mu_{k11}\left(\frac{\partial\phi_1}{\partial y}\mathrm{d}y^{(q-l-1)} + \frac{\partial\phi_1}{\partial u}\mathrm{d}u^{(q-l-1)}\right) \left(\bmod E^{q-l-1}\right)$$

$$\begin{aligned}
\mathrm{d}z_k^{(q-l-k)} &= \mu_{k21}\frac{\partial\phi_2}{\partial z_1}\mathrm{d}z_1^{(q-l-1)} \\
&\quad +\mu_{k21}\left(\frac{\partial\phi_1}{\partial y}\mathrm{d}y^{(q-l-1)} + \frac{\partial\phi_1}{\partial u}\mathrm{d}u^{(q-l-1)}\right) \left(\bmod E^{q-l-1}\right) \\
&\ \ \vdots \\
\mathrm{d}z_k^{(q-l-2)} &= \sum_{i=1}^{k-1}\frac{\partial\phi_k}{\partial z_i}\mathrm{d}z_i^{(q-l-1)} + \mu_{kk1}\sum_{i=1}^{k-2}\frac{\partial\phi_{k-1}}{\partial z_i}\mathrm{d}z_i^{(q-l-1)} + \cdots \\
&\quad +\mu_{kk(k-1)}\frac{\partial\phi_2}{\partial z_1}\mathrm{d}z_1^{(q-l-1)} + \mu_{k21}\left(\frac{\partial\phi_1}{\partial y}\mathrm{d}y^{(q-l-1)} + \frac{\partial\phi_1}{\partial u}\mathrm{d}u^{(q-l-1)}\right) \\
&\quad \left(\bmod E^{q-l-1}\right)
\end{aligned} \tag{12.45}$$

Define the space

$$\Omega_{l+1} = \mathrm{span}\{\sum_{i=1}^{k-1}\frac{\partial\phi_k}{\partial z_i}\mathrm{d}z_i, k=2,\dots,\min(l,q-l-1); \frac{\partial\phi_1}{\partial y}\mathrm{d}y + \frac{\partial\phi_1}{\partial u}\mathrm{d}u\}$$

from the definition of the differential one-form $\bar{\omega}_{l+1}$ in Step $l+1$ of Algorithm 12.9, and using (12.41) to (12.45), one concludes that the differential one-form $\pi_{l+1} \in \Omega_{l+1}$ such that

$$\begin{aligned}
\bar{\omega}_{l+1} - \pi_{l+1} &= \xi \cdot \frac{\partial\phi_n}{\partial\dot{\phi}_{n-1}}\cdots\frac{\partial\phi_{l+2}}{\partial\dot{\phi}_{l+1}}\left(\sum_{i=1}^{l}\frac{\partial\phi_{l+1}}{\partial z_i}\mathrm{d}z_i + \frac{\partial\phi_{l+1}}{\partial y}\mathrm{d}y + \frac{\partial\phi_{l+1}}{\partial u}\mathrm{d}u\right) \\
&= \bar{\xi}\mathrm{d}\phi_{l+1}
\end{aligned}$$

Then, from Lemma 12.8, the second check of Step $l+1$ is necessarily fulfilled.

The rest of the necessity can be proved in a similar vein. ∎

12.1.3 Example

Consider a system described by a higher order input-output differential equation

$$y^{(n)} = \varphi(y,\dots,y^{(n-1)},u,\dot{u},\dots,u^{(s)}) \tag{12.46}$$

Note that Theorems 12.1, 12.3, and 12.6 remain valid for system (12.46). However, any input-output system (12.46) that fulfills the conditions of Theorem 12.1 admits a state realization (12.1). An input-output system that fulfills the conditions of Theorem 12.3 does not necessarily admit a state realization.

Consider $y^{(4)} = \dot{y}^2\dot{u}+u+y\dot{u}+u\dot{y}+\dot{y}^2u+3y\dot{y}\ddot{u}+\ddot{y}y\dot{u}+y^2u^{(3)}+3\dot{y}y\dot{u}+y^2\ddot{u}+\ddot{y}uy$. This input-output system does not admit any standard state realization. Note that its relative degree is 1. Let $\omega = \mathrm{d}y^{(4)}$,

$$\begin{aligned}\omega = {} & y^2\mathrm{d}u^{(3)} + (3\dot{y}y + y^2)\mathrm{d}\ddot{u} + (y\dot{u} + yu)\mathrm{d}\ddot{y} \\ & +(\dot{y}^2 + 3\dot{y}y + y\ddot{y} + y)\mathrm{d}\dot{u} + (2\dot{y}\dot{u} + 3y\ddot{u} + 2\dot{y}u + 2\ddot{u}y + 3\dot{u}y + u)\mathrm{d}\dot{y} \\ & +(3\dot{y}\ddot{u} + \ddot{y}\dot{u} + 2yu^{(3)} + 3\dot{y}\dot{u} + 2y\ddot{u} + \ddot{y}u + \dot{u})\mathrm{d}y + (\dot{y}^2 + y\ddot{y} + \dot{y} + 1)\mathrm{d}u\end{aligned}$$

Step 1 of Algorithm 12.4 yields $\bar{\omega}_1 = y^2\mathrm{d}u$. Then, $\phi_1 = u$, $\lambda_1 = y^2$, and $z_2 = \dot{\phi}_1(y,u)$. ω is rewritten so that $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\ddot{z}_2,\ldots,\mathrm{d}z_2,\mathrm{d}\ddot{y},\ldots,\mathrm{d}y,\mathrm{d}u\}$:

$$\begin{aligned}\omega = {} & y^2\mathrm{d}\ddot{z}_2 + (3\dot{y}y + y^2)\mathrm{d}\dot{z}_2 + (yz_2 + yu)\mathrm{d}\ddot{y} \\ & +(\dot{y}^2 + 3\dot{y}y + y\ddot{y} + y)\mathrm{d}z_2 + (2\dot{y}z_2 + 3y\dot{z}_2 + 2\dot{y}u + 2\dot{z}_2y + 3z_2y + u)\mathrm{d}\dot{y} \\ & +(3\dot{y}\dot{z}_2 + \ddot{y}z_2 + 2y\ddot{z}_2 + 3\dot{y}z_2 + 2y\dot{z}_2 + \ddot{y}u + z_2)\mathrm{d}y + (\dot{y}^2 + y\ddot{y} + \dot{y} + 1)\mathrm{d}u\end{aligned}$$

Step 2 yields $\bar{\omega}_2 = y^2\mathrm{d}z_2 + yz_2\mathrm{d}y + yu\mathrm{d}y$. The relation $\mathrm{d}\bar{\omega}_2 \wedge \bar{\omega}_2 \wedge \mathrm{d}\phi_1 = 0$ is fulfilled. Then, $\frac{1}{y}\left(\bar{\omega}_2 + y^2\mathrm{d}u\right) = \mathrm{d}\phi_2 = \mathrm{d}(y\dot{\phi}_1 + uy)$, and $z_3 = \dfrac{\mathrm{d}}{\mathrm{d}t}(y\dot{u} + yu)$. ω is rewritten so that $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\dot{z}_3, \mathrm{d}z_3, \mathrm{d}\dot{y}, \mathrm{d}y, \mathrm{d}\dot{u}, \mathrm{d}u\}$:

$$\omega = y\mathrm{d}\dot{z}_3 + \dot{y}\mathrm{d}z_3 + (z_3 + u)\mathrm{d}\dot{y} + y\mathrm{d}\dot{u} + (\dot{z}_3 + \dot{u})\mathrm{d}y + (\dot{y} + 1)\mathrm{d}u$$

From Step 3, define $\bar{\omega}_3 = y\mathrm{d}z_3 + (z_3+u)\mathrm{d}y + y\mathrm{d}u$. The relation $\mathrm{d}\bar{\omega}_3 \wedge \bar{\omega}_3 \wedge \mathrm{d}\phi_2 = 0$ is fulfilled. Then, $\bar{\omega}_3 = \mathrm{d}\phi_3 = \mathrm{d}(y\dot{\phi}_2 + uy)$, and $z_4 = \dfrac{\mathrm{d}}{\mathrm{d}t}(y\dot{\phi}_2 + yu)$. ω is rewritten so that $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_4, \mathrm{d}y, \mathrm{d}u\}$: $\omega = \mathrm{d}z_4 + \mathrm{d}u$. Thus, $\mathrm{d}\phi_4 = \mathrm{d}\dot{\phi}_3 + \mathrm{d}u$.

A linearizing dynamic compensator is now obtained. From $\phi_1 = u$, $\phi_2 = y\dot{\phi}_1 + uy$, $\phi_3 = y\dot{\phi}_2 + uy$, and $\phi_4 = \dot{\phi}_3 + u$, set $\eta_1 = u$, $\eta_2 = y\dot{\eta}_1 + \eta_1 y$, $\eta_3 = y\dot{\eta}_2 + \eta_1 y$, and $v = \dot{\eta}_3 + \eta_1$. The compensator

$$\begin{aligned}\dot{\eta}_1 &= \tfrac{1}{y}(\eta_2 - \eta_1 y) & \dot{\eta}_3 &= v - \eta_1 \\ \dot{\eta}_2 &= \tfrac{1}{y}(\eta_3 - \eta_1 y) & u &= \eta_1\end{aligned}$$

linearizes the system, and the closed-loop system admits the following standard state realization:

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 & \dot{\eta}_1 &= \tfrac{1}{\zeta_1}(\eta_2 - \eta_1\zeta_1) \\ \dot{\zeta}_2 &= \zeta_3 & \dot{\eta}_2 &= \tfrac{1}{\zeta_1}(\eta_3 - \eta_1\zeta_1) \\ \dot{\zeta}_3 &= \zeta_4 & \dot{\eta}_3 &= v - \eta_1 \\ \dot{\zeta}_4 &= v & y &= \zeta_1\end{aligned}$$

12.1.4 The Hopping Robot

In this section, straightforward generalizations of the previous results are given for the multivariable case, and in the situation where the measured outputs

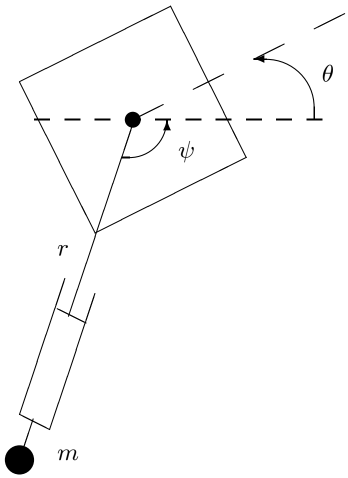

Fig. 12.2. Hopping robot

z are different from the outputs y to be controlled. Consider the kinematic model of a hopping robot in [61].

The input-output relation of this system can be linearized by dynamic measurement feedback. The system is not linearizable by output injections.

$$\begin{cases} \dot{\psi} = u_1 \\ \dot{\ell} = u_2 \\ \dot{\theta} = \frac{m(\ell+1)^2}{1+m(\ell+1)^2} u_1 \\ y_1 = \psi \\ y_2 = \theta \\ z = \ell \end{cases} \tag{12.47}$$

Compute $\ddot{y}_1 = \dot{u}_1$, $\ddot{y}_2 = \dfrac{2mu_1u_2(\ell+1)}{(1+m(\ell+1)^2)^2} + \dfrac{m\dot{u}_1(\ell+1)^2}{1+m(\ell+1)^2}$. Since

$$\ddot{y}_i = \phi_{i2}(\phi_{i1}, z, u) \ (i = 1, 2)$$

with $\phi_{11} = u_1$, $\phi_{12} = \dot{\phi}_{11}$,$\phi_{21} = u_1$, $\phi_{22} = \dfrac{2mu_1u_2(z+1)}{(1+m(z+1)^2)^2} + \dfrac{m\dot{\phi}_{21}(z+1)^2}{1+m(z+1)^2}$, there exists a dynamic measurement feedback which linearizes the input-output relation of (12.47). The compensator can be written as

$$\begin{aligned} u_1 &= \eta \\ \dot{\eta} &= v_1 \\ u_2 &= \left[v_2 + \frac{m(z+1)^2 v_1}{1+m(z+1)^2} \right] \frac{(1+m(z+1)^2)^2}{2\eta m(z+1)}. \end{aligned}$$

The closed-loop relations are $\ddot{y}_1 = v_1$ and $\ddot{y}_2 = v_2$.

12.2 Input-output Decoupling

Consider the square invertible system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{12.48}$$

where $x \in I\!R^n$, $u \in I\!R^m$, and $y \in I\!R^m$.

12.2.1 Input-output Decoupling via Static Output Feedback

Problem Statement

Given system (12.48), find if possible, a static output feedback $u = \alpha(y) + \beta(y)v$ such that

- $\mathrm{d}y_i^{(k)} \in \mathrm{span}\{\mathrm{d}x, \mathrm{d}u_i, ..., \mathrm{d}u_i^{(k-1)}, \forall k \geq 0$
- $\mathrm{d}y_i^{(k)} \notin \mathrm{span}\{\mathrm{d}x\}$

Example 12.12.

$$\begin{aligned} \dot{x}_1 &= x_3(\sin x_1 + u_1/x_2) \\ \dot{x}_2 &= x_3 + u_2 \\ \dot{x}_3 &= -x_3 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

This system can be decoupled without any feedback of x_3. The output feedback $u_1 = -y_2 \sin y_1$, $u_2 = v_2$ decouples the system.

To characterize fully the solvability of the problem, introduce a special subspace Ω_i associated with the output component y_i:

$$\Omega_i = \{\omega \in \mathcal{X} \mid \forall k \in I\!N, \omega^{(k)} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{(r_i)}, ..., \mathrm{d}y_i^{(r_i+k-1)}\}$$

The subspace Ω_i is a controllability cospace and characterizes those state variables that do not have to be rendered unobservable for noninteracting control. The main result is now in order.

Theorem 12.13. *System (12.48) can be decoupled by a static output feedback $u = \alpha(y) + \beta(y)v$ if and only if*

- *the relative degrees $r_1, ..., r_m$ are finite*
- $\dim(\mathcal{X} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1^{(r_1)}, ..., \mathrm{d}y_m^{(r_m)}\}) = n + m$
- $\mathrm{d}y_i^{(r_i)} \in \Omega_i \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j, \mathrm{d}u, j \neq i\}$
- $\mathrm{d}\pi_i \wedge \pi_i \wedge \mathrm{d}y_i = 0, \forall i = 1, ..., m,$ *where* $\pi_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j, \mathrm{d}u, j \neq i\}$ *is such that* $\mathrm{d}y_i^{(r_i)} - \pi_i \in \Omega_i$.

12.2.2 Input-output Decoupling Via Measurement Feedback

Problem Statement

Consider system (12.48) and the measurement $z = k(x)$, $z \in I\!R^q$. The input-output decoupling via static measurement feedback is solvable if there exists a feedback $u = \alpha(z) + \beta(z)v$ such that

- $\mathrm{d}y_i^{(j)} \in \mathrm{span}\{\mathrm{d}x, \mathrm{d}u_i, ..., \mathrm{d}u_i^{(j-1)}, \forall j \geq 0$
- $\mathrm{d}y_i^{(j)} \notin \mathrm{span}\{\mathrm{d}x\}$

Before giving the general solution of this problem, introduce the following notation. Given a subspace Ω of $\mathcal{X}$, the largest controlled invariant subspace contained in Ω is denoted Ω^*. Denote $\mathcal{Z} = \{\mathrm{d}z\}$.

Theorem 12.14. *System (12.48) is decouplable by measurement feedback if*

- *the relative degrees $r_1, ..., r_m$ are finite*
- $\dim(\mathcal{X} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1^{(r_1)}, ..., \mathrm{d}y_m^{(r_m)}\}) = n + m$
- $\mathrm{d}y_i(r_i) \in \Omega_i + \mathcal{Z} + \mathcal{U}$
- $\dim(\mathrm{span}_{\mathcal{K}}\{\pi_i\} \oplus \Omega_i \cap \mathcal{Z})^* = \dim(\Omega_i \cap \mathcal{Z})^* + 1$ *where $\pi_i \in \mathcal{Z} + \mathcal{U}$ is such that $\mathrm{d}y_i^{(r_i)} - \pi_i \in \Omega_i$, for $i = 1, ..., m$.*

Problem

12.1. Consider the virus dynamics model in Section 2.10.2

$$\begin{aligned} \dot{T} &= s - \delta T - \beta T v \\ \dot{T}^* &= \beta T v - \mu T^* \\ \dot{v} &= kT^* - cv \\ y_1 &= T \\ y_2 &= v \end{aligned} \tag{12.49}$$

Show that this model can be fully linearized by static output feedback.

References

1. Abraham R, Marsden JE, Ratiu T (1988) Manifolds, Tensor Analysis and Applications. Springer, New York
2. D'Andréa-Novel B (1988) Commande non linéaire des robots. Hermès Publishing, Paris
3. Aranda-Bricaire E, Moog CH, Pomet JB (1995) A linear algebraic framework for dynamic feedback linearization. IEEE Trans. Autom. Control 40:127–132
4. Aranda-Bricaire E, Kotta Ü, Moog CH (1996) Linearization of discrete-time systems. SIAM J. Control Optim. 34:1999–2023
5. Atihya MF, MacDonald IG (1969) Introduction to Commutative Algebra. Addison-Wesley Series in Mathematics
6. Basile G, Marro G (1992) Controlled and Conditioned Invariants in Linear System Theory. Prentice-Hall, Englewood Cliffs, NJ
7. Boothby WM (1986) An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd ed. (Pure and Applied Mathematics, vol 20) Academic Press, New York
8. Bornard G, Hammouri H (1991) A high-gain observer for a class of uniformly observable systems. In Proc. 30th IEEE Conf. Decision Control, Brighton, pp 1079–1090
9. Brockett RW (1978) Feedback invariants for nonlinear systems. In Proc. 7th IFAC World Congr., Helsinki, pp 1115–1120
10. Bryant RL, Chern SS, Gardner RB, Goldschmidt H L, Griffiths P A (1991) Exterior Differential Systems. (Mathematical Sciences Research Institute Publications, vol 18) Springer, New York
11. Byrnes CI, Isidori A (1988) Local stabilization of minimum phase nonlinear systems. Syst. Control Lett. 11:9–17
12. Byrnes CI, Isidori A (1990) Exact linearization and zero dynamics. In Proc. 29th IEEE Conf. Decision Control, pp 2080–2084
13. Castro-Linares R, Moog CH, Velasco Villa M (1997) Bi-Causal solutions to the disturbance decoupling problem for time-delay nonlinear systems. In Proc. 36th IEEE Conf. Decision Control, San Diego
14. Cao L, Zheng YF (1992) On minimal compensators for decoupling control. Syst. Control Lett. 18:121–128
15. Charlet B, Lévine J, Marino R (1989) On dynamic feedback linearization. Syst. Control Lett. 13:143–151

16. Charlet B, Lévine J, Marino R (1991) Sufficient conditions for dynamic feedback linearization. SIAM J. Control Optim. 29:38–57
17. Cheng D, Isidori A, Respondek W, Tarn TJ (1988) Exact linearization of nonlinear systems with outputs. Math. Syst. Theory. 21:63–83
18. Chiasson J, Nonlinear differential-geometric techniques for control of a series DC motor. IEEE Trans. Control Syst. Tech. 2:35–42
19. Choquet-Bruhat Y, DeWitt-Morette C, Dillard-Bleick M (1989) Analysis, Manifolds and Physics, Part I: Basics. North-Holland, Amsterdam
20. Claude D (1986) Everything you always wanted to know about linearization. In: Fliess M, Hazewinkel M (eds) Algebraic and Geometric Methods in Nonlinear Control Theory. Reidel, Dordrecht, pp 181–220
21. Conte G, Moog CH, Perdon AM (1988) Un théorème sur la représentation entrée-sortie d'un système non linéaire. C. R. Acad. Sci. Paris. 307-I:363–366
22. Conte G, Moog CH, Perdon AM (1991) Model matching and factorization for nonlinear systems: a structural approach. SIAM J. Control Optim. 29:769–785
23. Conte G, Moog CH, Perdon AM (1999) Nonlinear Control Systems: an Algebraic Setting. (Lecture Notes on Control and Information Sciences, vol 242) Springer Verlag, London, 1999
24. Conte G, Perdon AM (1985) On the causal factorization problem. IEEE Trans. Autom. Control 30:811–813
25. Conte G, Perdon AM (1985) Zero modules and factorization problems. Contemp. Math. 47:81–94
26. Conte G, Perdon AM , Moog CH (1993) The differential field associated to a general analytic nonlinear system. IEEE Trans. Autom. Contr. 38:1120–1124.
27. Crouch PE, Lamnabhi-Lagarrigue F (1988) State-space realization of nonlinear systems defined by input-output differential equations. In: Bensoussan A, Lions JL (eds) Analysis and Optimization Systems. (Lecture Notes on Control and Information Sciences, vol. 111) Springer, Berlin Heidelberg New York, pp 138–149
28. Crouch PE, Lamnabhi-Lagarrigue F (1992) Realizations of input-output differential equations. In: Kimura H, Kodama S (eds) Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing II. Mita Press, Tokyo, pp 259–264
29. Crouch PE, Lamnabhi-Lagarrigue F, Pinchon D (1995) Some realization algorithms for nonlinear input-output systems. Int. J. Control 62:941–960
30. Delaleau E (1993) Sur les dérivées de l'entrée en représentation et commande des systèmes des non linéaires. Ph.D. Thesis, Univ. Paris XI
31. Delaleau E (1995) Sur la synthèse par bouclage dynamique. In: Proc. Seminar "Développements récents en non linéaire", LSS-CNRS, Paris, pp 9–23
32. Delaleau E, Fliess M (1992) An algebraic interpration of the structure algorithm with an application to feedback decoupling. In: Proc. Nonlinear Control Syst. Design IFAC Symp., Bordeaux, pp 489–494
33. Delaleau E, Respondek W (1995) Lowering the orders of derivatives of controls in generalized state space systems (summary). J. Math. Syst. Estim. Control 5:375–378
34. Delaleau E, Pereira da Silva P (1994) Rank conditions for the dynamic disturbance decoupling problem. In: Proc. 33rd IEEE Conf. Decision Control, Buena Vista, pp 1784–1789
35. Delaleau E, Pereira da Silva P (1998) Filtrations in feedback synthesis: Part I - Systems and feedbacks. Forum Mathematicum 10:147–174

36. Delaleau E, Pereira da Silva P (1998) Filtrations in feedback synthesis: Part II - Input-output decoupling and disturbance decoupling. Forum Mathematicum. 10:259–276.
37. Descusse J, Moog CH (1985) Decoupling with ynamic compensation for strong invertible affine non-linear systems. Int. J. Control 42:1387–1398
38. Descusse J, Moog CH (1987) Dynamic decoupling for right-invertible nonlinear systems. Syst. Control Lett. 8:345–349
39. Di Benedetto MD (1989) A condition for the solvability of the nonlinear model matching problem. In: Descusse J, Fliess M, Isidori A, Leborgne D (eds)New Trends in Nonlinear Control Theory. (Lecture Notes in Control and Information Sciences, vol 122) Springer Verlag, Berlin, pp 102–115
40. Di Benedetto MD, Glumineau A, Moog CH (1991) Equivalence of nonlinear systems under dynamic state feedback. In: Proc. 30th IEEE Conf. Decision Control, Brighton, pp 400–405
41. Di Benedetto MD, Glumineau A, Moog CH (1994) The nonlinear interactor and its application to input-output decoupling. IEEE Trans. Autom. Control 39:1246–1250
42. Di Benedetto MD, Grizzle JW, Moog CH (1989) Rank invariants of nonlinear systems. SIAM J. Control Optim. 27:658–672
43. Di Benedetto MD, Isidori A (1986) The matching of nonlinear models via dynamic state-feedback. SIAM J. Control Optim. 24:1063–1075
44. Diop S (1991) Elimination in control theory. Math. Control Signals Syst. 4:72–86
45. Diop S, Fliess M (1991) On nonlinear observability. In: Proc. Eur. Control Conf., Grenoble, pp 152–157
46. Flanders H (1963) Differential Forms with Application to the Physical Sciences. (Mathematics in Science and Engineering, vol. 11) Academic Press, New York
47. Fliess M (1980) Realizations of nonlinear systems and abstract transitive Lie algebras. Bull. Am. Math. Soc. 2:444–446
48. Fliess M (1981) Fonctionnelles causales non linéaires et indéterminées non commutatives. Bull. Soc. Math. France. 109:3–40
49. Fliess M (1985) A new approach to the noninteracting control problem in nonlinear systems theory. In: Proc. 23rd Allerton Conf., Monticello, pp 123–129
50. Fliess M (1985) Some remarks on nonlinear invertibility and dynamic state feedback. In: Byrnes C, Lindquist A (eds) Proc. MTNS'85. North-Holland, Amsterdam
51. Fliess M (1986) A note on the invertibility of nonlinear input-output systems. Syst. Control Lett. 8:147–151.
52. Fliess M (1989) Automatique et corps différentiels. Forum Mathematicum. 1:227–238
53. Fliess M (1986) Automatique en temps discret et algèbre aux différences. Forum Mathematicum. 2:213–232
54. Fliess M (1990) Generalized controller canonical forms for linear and nonlinear dynamics. IEEE Trans. Autom. Control 35:994–1001
55. Fliess M (1993) Some remarks on the Brunovsky canonical form. Kybernetika. 29:417–422
56. Fliess M, Glad ST (1993) An algebraic approach to linear and nonlinear control. In: Essays on Control: Perspectives in the Theory and its Applications. Birkhäuser, Boston

57. Fliess M, Lévine J, Martin P, Rouchon P (1992) Sur les systèmes non linéaires différentiellement plats. C. R. Acad. Sc. Paris. 315-I:619–624
58. Fliess M, Lévine J, Martin P, Rouchon P (1992) On differentially flat nonlinear systems. In: Proc. 2nd IFAC NOLCOS Symp., Bordeaux, pp 408–412
59. Fliess M, Lévine J, Martin P, Rouchon P (1993) Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. C. R. Acad. Sc. Paris. 317-I:981–986
60. Fliess M, Lévine J, Martin P, Rouchon P (1993) Towards a new differential geometric setting in nonlinear control. In: International Geometrical Colloquium, Moscow
61. Fliess M, Lévine J, Martin P, Rouchon P (1996) Systèmes différentiellement plats : intérêt et exemples. In: Gentil S (ed) Théorie et pratique des systèmes plats. Laboratoire d'Automatique de Grenoble, pp 1-41
62. Gardner RB, Shadwick WF (1992) The GS algorithm for exact linearization to Brunovsky normal form. IEEE Trans. Autom. Control 37:224–230
63. Gauthier JP, Kupka I (1994) Observability and observers for nonlinear systems. SIAM J. Control Optim. 32:975–994
64. Glad S (1989) Nonlinear state space and input-output description using differential polynomials. In: Descusse J, Fliess M, Isidori A, Leborgne D (eds) New Trends in Nonlinear Control Theory. (Lecture Notes in Control and Information Science, vol 122) Springer Verlag, Berlin, pp 182–189
65. Glumineau A, Moog CH (1989) Essential orders and the nonlinear decoupling problem. Int. J. Control 50:1825–1834
66. Godbillon C (1969) Géométrie Différentielle et Mécanique Analytique. Hermann, Paris
67. Grizzle JW (1986) Feedback linearization of discrete-time systems. In: Bensoussan A, Lions JL (eds) Analysis and Optimization Systems. (Lecture Notes in Control and Information Sciences, vol 83) Springer, Heidelberg, pp 273–281
68. Grizzle JW (1993) A linear algebraic framework for the analysis of discrete-time nonlinear systems. SIAM. J. Control Optim. 31:1026–1044
69. Haddak A (1990) Contribution à l'étude de la théorie du contrôle aux dérivées partielles. Ph.D. Thesis, ENPC, Paris
70. Hammer J, Heymann M (1981) Causal factorization and linear feedback. SIAM J. Control Optim. 19:445–468
71. Hammouri H, Gauthier JP (1988) Bilinearization up to output injection. Syst. Control Lett. 11:139–149
72. Hammouri H, Gauthier JP (1992) Global time-varying linearization up to output injection. SIAM. J. Control Optim. 30:1295–1310
73. Herman R, Krener AJ (1977) A nonlinear controllability and observability. IEEE. Trans. Autom. Control 22:728–740
74. Hirschorn R (1979) Invertibility of multivariable nonlinear control systems. IEEE Trans. Autom. Control 24:855–865
75. Hirschorn RM (1981) (A,B)-invariant distributions and disturbance decoupling of nonlinear systems. SIAM. J. Control Optim. 19:1–19
76. Huijberts HJC (1991) Dynamic feedback in nonlinear synthesis problems. Ph. D. Thesis, University of Twente.
77. Huijberts HJC (1992) A nonregular solution of the nonlinear dynamic disturbance decoupling problem with an application to a complete solution of the nonlinear model matching problem. SIAM. J. Control Optim. 30:350–366

78. Huijberts HJC, Nijmeijer H (1990) Local nonlinear model matching: from linearity to nonlinearity. Automatica 26:173–183
79. Huijberts HJC, Nijmeijer H, van der Wegen LLM (1992) Minimality of dynamic input-output decoupling for nonlinear systems. Syst. Control Lett. 18:435–443
80. Huijberts HJC, Nijmeijer H, van der Wegen LLM (1992) Dynamic disturbance decoupling for nonlinear systems. SIAM. J. Control Optim. 30:336–349
81. Huijberts HJC, Moog CH, Pothin R (2000) Input-output decoupling of nonlinear systems by static measurement feedback. Syst. Control Lett. 39:109–114
82. Hunt LR, Su R, Meyer G (1983) Design for multi-input nonlinear systems. In: Brockett R, Millmann R, Sussmann H (eds) Differential Geometric Control Theory. Birkhäuser, Boston, pp 268–298
83. Isidori A (1984) Nonlinear feedback, structure at infinity and the input-output linearization problem. In: In: Fuhrmann PA (ed) Mathematical Theory of Networks and Systems. Proc. MTNS-83 Beer Sheva. (Lecture Notes on Control and Information Sciences, vol 58) Springer Verlag, Heidelberg, pp 473–493
84. Isidori A (1985) The matching of a prescribed linear input-output behavior in a Nonlinear System. IEEE. Trans. Autom. Control 30:258–265
85. Isidori A (1986) Control of nonlinear systems via dynamic state-feedback. In: Fliess M, Hazewinkel M (eds) Algebraic and Geometric Methods in Nonlinear Control Theory. Reidel, Dordrecht
86. Isidori A (1989) Nonlinear Control Systems, 2nd ed. (Communications and Control Engineering Series) Springer, Berlin
87. Isidori A, Krener AJ, Gori-Giorgi C, Monaco S (1981) Nonlinear decoupling via feedback: a differential geometric approach. IEEE. Trans. Autom. Control 26:331–345.
88. Isidori A, Moog CH (1988) On the nonlinear equivalent of the notion of transmission zeros. In: Byrnes CI, Kurzhanski A (eds) Modeling and Adaptive Control. (Lecture Notes in Control and Information Sciences, vol 105) Springer, Berlin, pp 146–158
89. Isidori A, Moog CH, De Luca A (1986) A sufficient condition for full linearization via dynamic state feedback. In: Proc. 25th IEEE Conf. Decision Control, pp 203–208
90. Isidori A, Ruberti A (1984) On the synthesis of linear input-output responses for nonlinear systems. Syst. Control Lett. 4:17–22
91. Jakubczyk B (1986) Local realizations of nonlinear causal operators. SIAM J. Control and Optim. 24:231–242
92. Jakubczyk B (1986) Realization theory for nonlinear systems; three approaches. In: Fliess M, Hazewinkel M (eds) Algebraic and Geometric Methods in Nonlinear Control Theory. Reidel, Dordrecht, pp 3–13
93. Jakubczyk B (1987) Feedback linearization of discrete-time systems. Syst. Control Lett. 9:411–416
94. Jakubczyk B (1992) Remarks on equivalence and linearization of nonlinear systems. In: Proc. 2nd. IFAC NOLCOS Symp., Bordeaux, pp 393–397
95. Jakubczyk B, Normand-CyrotD (1984) Orbites de pseudo-groupes de diffeomorphismes et commandabilité des systèmes non linéaires en temps discret. C. R. Acad. Sci. Paris. 298:257–260
96. Jacubczyk B, Respondek W (1980) On linearization of control systems. Bull. Acad. Pol. Sci., Ser. Sci. Math. 18:517–522
97. Jakubczyk B, Sontag E D (1990) Controllability of nonlinear discrete-time systems: a Lie-algebraic approach. SIAM J. Control Optim. 28:1–33

98. Kaplansky I (1976) An Introduction to Differential Algebra, 2nd ed. Hermann, Paris
99. Kalman RE, Falb PL, Arbib MA (1969) Topics in Mathematical System Theory. McGraw-Hill, New York
100. Kawski M (1991) High-order conditions for local controllability in practice. In: Kimura H, Kodama S (eds) Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing. Mita Press, Tokyo, pp 271–276
101. Khalil HK (1992) Nonlinear Systems. Macmillan, New York
102. Kolchin ER (1973) Differential Algebra and Algebraic Groups. Academic Press, New York
103. Kotta Ü (1995) Inversion Method in the Discrete-Time Nonlinear Control Systems Synthesis Problems. (Lecture Notes in Control and Information Sciences, vol 205) Springer Verlag, London
104. Krener AJ, Isidori A (1983) Linearization by output injection and nonlinear observers. Syst. Control Lett. 3:47–52.
105. Krener AJ, Respondek W (1985) Nonlinear observers with linearizable error dynamics. SIAM J. Control Optim. 23:197-216
106. Krener AJ, Isidori A, Respondek W (1983) Partial and robust linearization by Feedback. In: Proc. 22nd IEEE Conf. Decision Control, pp 126–130
107. Kung S, Kailath I (1978) Some notes on valuation theory in linear systems. In: Proc. 17th IEEE Conf. Decision Control, pp 515–517
108. Lee HG, Arapostatis A, Marcus SI (1987) Linearization of discrete-time systems. Int. J. Control. 45:1803–1882
109. Lee HG, Marcus SI (1986) Approximate and local linearization of nonlinear discrete-time systems. Int. J. Control. 44:1103–1124
110. Lévine J, Marino R (1990) On dynamic feedback linearization on $\mathbf{R}^4$. In: Proc. 29th IEEE Conf. Decision Control, pp 2088–2090
111. Lobry C (1970) Contrôlabilité des Systèmes Non Linéaires. SIAM J. Control 8:573–605
112. Marino R (1986) On the largest feedback linearizable subsystem. Syst. Control Lett. 7:345–351
113. Marino R, Tomei P (1991) Global adaptive observers and output-feedback stabilization for a class of nonlinear systems. In: Kokotovic P V (ed) Foundations of Adaptive Control. Springer-Verlag, Berlin, pp 455-493
114. Marino R, Tomei P (1995) Nonlinear Control Design: Geometric, Adaptive and Robust. Prentice-Hall, New York
115. Martin P (1992) Contribution à l'étude des systèmes non linéaires différentiellement plats. PhD thesis, Ecole des Mines de Paris
116. Martin P (1993) A geometric sufficient condition for flatness of systems with m inputs and $m+1$ states. In: Proc. 32nd IEEE Conf. Decision Control, San Antonio, pp 3431–3436
117. Martin P, Murray RM, Rouchon P (1997) Flat systems. In: Bastin G, Gevers M (eds) Plenary Lectures and Mini-courses, ECC 97, CIACO, Louvain-la-Neuve
118. Matsushima Y (1972) Differential Manifolds. (Pure and Applied Mathematics vol 9) Marcel Dekker, New York
119. Moog CH (1988) Nonlinear decoupling and structure at infinity. Math. Control Signals Systems 1:257–268

120. Moog CH, Castro Linares R, Velasco Villa M (1996) The dynamic disturbance decoupling problem for time-delay nonlinear systems. In: Proc. 35th IEEE Conf. Decision Control, Kobe, pp 821-822
121. Moog CH, Perraud J, Bentz P, Vo Q (1989) Prime differential ideals in nonlinear rational control systems. In: Proc. NOLCOS 89, Capri, pp 178–182
122. Moog CH, Plestan F, Perdon A M, Conte G (1993) On canonical forms of nonlinear systems. In: Proc. ECC 93, Groningen, pp 1514-1517
123. Morse AS (1973) Structural invariants of linear multivariable systems. SIAM J. Control 11:446–465
124. Nam K (1989) Linearization of discrete-time nonlinear systems and a canonical structure. IEEE Trans. Autom. Control 34:119–121
125. Nijmeijer H, Respondek W (1988) Dynamic input-output decoupling of nonlinear control systems. IEEE Trans. Autom. Control 33:1065–1070
126. Nijmeijer H, van der Schaft A (1990) Nonlinear Dynamical Control Systems. Springer, New York
127. Nijmeijer H, Schumacher JM (1985) On the inherent integration structure of nonlinear systems. IMA J. Math. Control Inf. 2:87–107
128. Nijmeijer H, Schumacher JM (1986) The regular local noninteracting control problem for nonlinear control systems. SIAM J. Control Optim. 24:1232–1245
129. Nowak MA, May RM (2002) Virus Dynamics: Mathematical Principles of Immunology and Virology. Oxford University Press, Oxford
130. Okutami T, Furuta K (1984) Model matching of nonlinear systems. In: Gertler J, Keviczky L (eds) Proc. IFAC 9th World Congr., Vol. IX, Budapest, pp 168–172
131. Plestan F (1995) Linéarisation par injection d'entrée-sortie généralisée et synthèse d'observateurs. PhD Thesis, Univ. Nantes/Ecole Centrale Nantes
132. Perdon AM, Conte G, Moog CH (1990) Some canonical properties of nonlinear systems. In: Kaashoek M, van Schuppen J, Ran A (eds) Realization and Modelling in System Theory, Birkhäuser, Cambridge, MA, pp 89–96
133. Polderman JW, Willems JC (1988) Introduction to Mathematical Systems Theory: A Behavioral Approach. Springer Verlag, New York
134. Perdon AM, Zheng YF, Moog CH, Conte G (1993) Disturbance decoupling for nonlinear systems : a unified approach. Kybernetika 29:479–484
135. Pomet JB (1997) On dynamic feedback linearization of four-dimensional affine control systems with two inputs. ESAIM-COCV. 2:151–230.
136. Pommaret JF (1986) Géométrie différentielle algébrique et théorie du contrôle. C. R. Acad. Sci. Paris, Série I. 302:547–550
137. Respondek W (1986) Partial linearization, decompositions and fibre linear systems. In: Byrnes C I, Lindquist A (eds) Theory and Applications of Nonlinear Control Systems. North-Holland, Amsterdam, pp 137–154
138. Respondek W (1993) Dynamic input-output linearization and decoupling of nonlinear systems. In: Proc. 2nd Eur. Control Conf., Groningen, pp 1523–1527
139. Respondek W (1994) Personal communication
140. Ritt JF (1950) Differential Algebra, American Mathematical Society, vol 33, Providence, RI ($http://www.ams.org/online\_bks/coll33/$)
141. Rubel L, Singer M (1985) A differentially algebraic elimination theorem with applications to analog computations in the calculus of variations. Proc. A.M.S. 94:653–658
142. Rudolph J (1991) Poursuite de modèle: une approche par l'algèbre différentielle. PhD Thesis, Université de Paris XI Orsay

143. Sain M K, Massey JL (1969) Invertibility of linear time-invariant dynamical systems. IEEE Trans. Autom. Control 14:141–149
144. Silverman LM (1969) Inversion of multivariable linear systems. IEEE Trans. Autom. Control 14:270–276
145. Singh SN (1981) A modified algorithm for invertibility in nonlinear systems. IEEE Trans. Autom. Control 26:595–598
146. Slotine JJE, Hedrick JK, Misawa EA (1987) On sliding observers for nonlinear systems. A.S.M.E. J. Dynamic Syst. Meas. Control 109:245–252
147. Slotine JJE, Li W (1991) Applied nonlinear control. Prentice-Hall, Englewood Cliffs, NJ
148. Sluis WM (1992) Absolute equivalence and its applications to control theory. Ph. D. Thesis, University of Waterloo
149. Sluis WM (1993) A necessary condition for dynamic feedback linearization. Syst. Control Lett. 21:277–283
150. Sontag ED (1988) Bilinear realizability is equivalent to existence of a singular affine differential i/o equation. Syst. Control Lett. 11:181–187
151. Sontag ED, Wang Y (1990) Input/output equations and realizability. In: Kaashoek M, van Schuppen J, Ran A (eds) Realization and Modelling in System Theory. Birkhäuser, Cambridge, MA, pp 125–132
152. Spivak M (1979) A Comprehensive Introduction to Differential Geometry. Publish or Perish, Houston
153. Su R (1982) On the linear equivalents of nonlinear systems. Syst. Control Lett. 2:48–52
154. Sussmann H, Jurdjevic V (1972) Controllability of nonlinear systems. J. Diff. Eqs. 12:95–116
155. Sussmann H (1973) Orbits of families of vector fields and integrability of distributions. Trans. Am. Math. Soc. 180:171–188
156. van der Schaft AJ (1989) Representing a nonlinear state space system as a set of higher order differential equations in the inputs and outputs. Syst. Control Lett. 12:151–160
157. Vidyasagar M (1993) Nonlinear systems analysis, 2nd ed. Prentice-Hall, Englewood Cliffs, NJ
158. Wang Y, Sontag ED (1992) Generating series and nonlinear systems: analytic aspects, local realizability, and i/o representations. Forum Mathematicum 4:299–322
159. Willems JC (1991) Paradigms and puzzles in the theory of dynamical systems. IEEE Trans. Autom. Control 36:259–294
160. Wonham WM (1985) Linear Multivariable Control: A Geometric Approach, 2nd ed. (Applications of Mathematics, vol 10) Springer Verlag, New York
161. Xia X, Gao WB (1989) Nonlinear observers design by dynamic error linearization. SIAM J. Control Optim. 27:199-216
162. Xia X, Moog CH (1996) Input output linearization of nonlinear systems by output feedback. In: Proc. 35th IEEE Conf. Decision Control, Vol 1, pp 82 – 83
163. Zeitz M (1985) Canonical forms for nonlinear systems. In: Jakubcyk B, Respondek W, Tchon K (eds) Geometric Theory of Nonlinear Control Systems. Tech. Univ. of Wroclaw, Poland.
164. Zheng YF (1993) Absolute observabilities of nonlinear control systems and their normal forms. Acta Mathematicæ Applicatæ Sinica 9:28–35

165. Zheng YF, Cao L (1993) On general construction of reduced inverse for controlled systems. Math. Control Signals Syst 6: 363-379
166. *http://www.engin.umich.edu/group/ctm/examples/ball/ball.html*

Index